Athens Institute
Working Paper No. 2025-2760-05
14 January 2025

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This paper should be cited as follows:

Bose, Babhrubahan, (2025) Point-wise Symmetry of Birkhoff-James Orthogonality and Geometry of $\mathbb{B}(\ell_\infty^n, \ell_1^m)$. Published by the Athens Institute: Working Paper No. 2025-2760-05, 14 January 2025. Pages 1-12

No.: 2025-2760-05
Date: 14 January 2025
DOI:
ISSN: 2241-2891

Previous Working Papers available at: www.atiner.gr/papers.htm

This series began in 2012 and was known as the Conference Paper Series until 2024. In 2025, the series was renamed and is now called the Working Paper Series.

Athens Institute (www.atiner.gr)
2025

Athens Institute's Working Paper Series No. 2025-2760-05, 14 January 2025, pages: 1-11

Point-wise Symmetry of Birkhoff-James Orthogonality and Geometry of $\mathbb{B}(\boldsymbol{\ell}_{\infty}^{\boldsymbol{n}}, \boldsymbol{\ell}_{\mathbf{1}}^{\boldsymbol{m}})$

By Babhrubahan Bose[*]

We study the relationship between the pointwise symmetry of Birkhoff-James orthogonality and the geometry of the space of operators $\mathbb{B}(\ell_{\infty}^{n}, \ell_{1}^{m})$*. We show that any non-zero left-symmetric point in this space is a smooth point. We also show that for* $n \geq 4$*, any unit norm right-symmetric point of this space is an extreme point of the closed unit ball. This marks the first step towards characterizing the extreme points of these unit balls and finding the Grothendieck constants* $G^{\mathbb{R}}(m,n)$ *using Birkhoff-James orthogonality techniques.*

Keywords: *Birkhoff-James orthogonality, extreme points, smooth points, left-symmetric points, right-symmetric points, Grothendieck constants*

2020 AMS Subject Classification: *Primary 46B20, Secondary 46B28, 46A32*

Introduction

In recent times, Birkhoff-James orthogonality and its pointwise symmetry has been used to understand the geometry of a normed linear space. Characterization of Birkhoff-James orthogonality and its local symmetry has been done for finite-dimensional ℓ_p spaces in [6], while that for the sequence spaces ℓ_p and the function spaces L_p have been done in [4] and [5]. In these articles, these characterizations have been used to understand the geometry of the underlying spaces by describing the smooth points and the onto isometries of the spaces. In this article, we use this idea to establish a relationship between the point-wise symmetry of Birkhoff-James orthogonality and the geometry of the space of operators from ℓ_{∞}^{n} into ℓ_{1}^{m} over $\mathbb{R}$ denoted by $\mathbb{B}(\ell_{\infty}^{n}, \ell_{1}^{m})$. We show that any non-zero left-symmetric point of this space is a smooth point and any unit norm right-symmetric point of this space is an extreme point of the closed unit ball.

Recall that these extreme points play a crucial role in understanding the *Grothendieck constant* for a given pair of natural numbers (m,n) given by

$$G^{\mathbb{R}}(m,n) := \sup_{\|T\|_{op}=1} \left|\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \langle x_i, y_j \rangle\right|, (1)$$

where $\|x_i\|_{\ell_2^d} = \|y_j\|_{\ell_2^d} = 1$, $T = [a_{ij}]_{m \times n} \in \mathbb{B}(\ell_{\infty}^{n}, \ell_{1}^{m})$ over $\mathbb{R}$. The Grothendieck constant $G^{\mathbb{R}}$ for $\mathbb{R}$ can then be obtained as the supremum of all $G^{\mathbb{R}}(m,n)$, where m and n vary over the natural numbers.

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The Grothendieck constant emerged from a celebrated result by Grothendieck [7], which in effect says that $G(m,n)$ as defined in (1) is uniformly bounded over both $\mathbb{R}$ and $\mathbb{C}$. This theorem/inequality of Grothendieck, including determining the exact constant (or sharper bounds for it [12]), has been the focus of tremendous research for the past few decades, including in Banach space theory, C^* algebra theory, operator theory, physics, and computer science (see e.g., the survey [11]). We refer the reader to the comprehensive and authoritative survey by Pisier [14] and the memoir by Blei [3] for more on the Grothendieck inequality.

The goal of this short note is the approach the Grothendieck inequality via the framework of Birkhoff-James orthogonality. Since the right-hand side of (1) is the supremum of a convex function taken over a convex set, the supremum is attained at the extreme points of the convex set, viz., the closed unit ball of $\mathbb{B}(\ell_\infty^n, \ell_1^m)$. Hence characterization of these extreme point, which we call the *extreme contractions* (following [8]), would allow us to find the Grothendieck constant for the given pair (m,n).

In this spirit, the first step towards better bounding the Grothendieck constant would be to characterize the extreme contractions. This has been done for $m = 1$, $n = 1$, and $(m,n) = (2,2), (3,3)$ in [1] and [13]. In this note we focus on the case $n \geq 4$, where our main result provides a class of extreme contractions:

Theorem 1. *Let* $n \geq 4$. *If* $T \in \mathbb{B}(\ell_\infty^n, \ell_1^m)$ *is a right-symmetric point and has norm 1, then* T *is an extreme contraction.*

For proving this result, we first study Birkhoff-James orthogonality in the Banach space $\ell_\infty^n \otimes \mathbb{X}$ for any Banach space $\mathbb{X}$. Using the results obtained therefrom, we prove the relationship between the point-wise symmetry of Birkhoff-James orthogonality and the geometry of the space of operators mentioned before.

Notations and Terminologies

Let us establish the relevant notations and terminologies to be used throughout the article. All the Banach spaces considered here are over $\mathbb{R}$. For a Banach space $\mathbb{X}$, let $\mathbb{X}^*$ denote the continuous dual of it and define the support functional of a non-zero element $x \in \mathbb{X}$ to be any $f \in \mathbb{X}^*$ such that

$$\|f\| = 1, f(x) = \|x\|.$$

A non-zero element $x \in \mathbb{X}$ is said to be smooth if it has a unique support functional.

Given two elements $x, y \in \mathbb{X}$, x is defined to be *Birkhoff-James orthogonal* to y [2], denoted by $x \perp_B y$ if

$$\|x + \lambda y\| \geq \|x\|, for\ every\ scalar\ \lambda.$$

James proved in [10] that $x \perp_B y$ if and only if $x = 0$ or $f(y) = 0$ for some support functional f of x. In the same article, he proved that a non-zero point $x \in \mathbb{X}$ is smooth if and only if Birkhoff-James orthogonality is right-additive at x, i.e., for any $y, z \in \mathbb{X}$,

$$x \perp_B y, x \perp_B z \Rightarrow x \perp_B (y + z).$$

James proved in [9] that in a normed linear space of dimension 3 or more, Birkhoff-James orthogonality is symmetric if and only if the space is an inner product space. However, the importance of studying the point-wise symmetry of Birkhoff-James orthogonality in describing the geometry of normed linear spaces has been illustrated in [6, Theorem 2.11], [17, Corollary 2.3.4]. Let us recall the following definition in this context from [16], which will play an important part in our present study.

Definition 2. *An element x of a normed linear space $\mathbb{X}$ is said to be left-symmetric (resp. right-symmetric) if*

$$x \perp_B y \Rightarrow y \perp_B x \;\; (resp.\; y \perp_B x \Rightarrow x \perp_B y),$$

for every $y \in \mathbb{X}$.

Note that by the term *point-wise symmetry of Birkhoff-James orthogonality*, we refer to the left-symmetric and the right-symmetric points of a given normed linear space.

A *semi-inner product* on a real vector space $\mathbb{V}$ is defined to be a map $[.,.]: \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ such that for $x, y, z \in \mathbb{V}$ and $\lambda \in \mathbb{R}$,

(1) $[x, x] \geq 0$ with equality if and only if $x = 0$.
(2) $[x, y] + \lambda[x, z] = [x, y + \lambda z]$.
(3) $[\lambda x, y] = \lambda[x, y]$.
(4) $[x, y]^2 \leq [x, x][y, y]$.

A *semi-inner product* on a Banach space $\mathbb{X}$ is a map $[.,.]: \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ satisfying the above four properties along with $[x, x] = \|x\|^2$ for every $x \in \mathbb{X}$. Construction of a semi-inner product on $\mathbb{X}$ requires a map $\Psi: S_{\mathbb{X}} \to \mathbb{X}^*$ such that $\Psi(x)$ is a support functional of x and $\Psi(-x) = -\Psi(x)$ for every $x \in S_{\mathbb{X}}$. The semi-inner product can then be constructed as $[0, y] = 0$ and

$$[x, y] = \|x\| \left(\Psi\left(\frac{x}{\|x\|} \right) \right)(y), \qquad x, y \in \mathbb{X}, x \neq 0.$$

Note that a non-zero point $x \in \mathbb{X}$ is smooth if and only if

$$[x, y]_1 = [x, y]_2 \; for \; every \; y \in \mathbb{X},$$

where $[.,.]_1$ and $[.,.]_2$ are any two semi-inner products on $\mathbb{X}$. Also for $x, y \in \mathbb{X}$, $x \perp_B y$ if and only if $[x, y] = 0$ for some semi-inner product $[.,.]$ on $\mathbb{X}$.

Let us denote the space $\ell_\infty^n \otimes \mathbb{X}$ by $\mathbb{X}_\infty^n$. Then $\mathbb{X}_\infty^n$ is the vector space of all $\mathbb{X}$-valued sequences of length n with the norm defined as

$$\|(x_1, x_2, \dots, x_n)\| := \max_{1 \le i \le n \|x_i\|} \|x_i\|, x_i \in \mathbb{X}.$$

Also denote $\ell_1^n \otimes \mathbb{X}$ by $\mathbb{X}_1^n$, i.e.,

$$\mathbb{X}_1^n := \{(x_1, x_2, \dots, x_n) : x_i \in \mathbb{X}\}, \|(x_1, x_2, \dots, x_n)\| := \sum_{i=1}^{n} \|x_i\|, x_i \in \mathbb{X}.$$

The Banach space of all bounded linear maps between two Banach spaces $\mathbb{X}$ and $\mathbb{Y}$ equipped with the operator norm is denoted by $\mathbb{B}(\mathbb{X}, \mathbb{Y})$. We call an extreme point of the closed unit ball of this space, *extreme contraction*. Also, the extreme points of the closed unit ball of any Banach space $\mathbb{X}$ are denoted simply by *extreme points of* $\mathbb{X}$. An operator $T \in \mathbb{B}(\mathbb{X}, \mathbb{Y})$ is said to attain norm at $x \in \mathbb{X}$ if $\|x\| = 1$ and $\|Tx\| = \|T\|$. The set of all points where an operator T attains norm is denoted by M_T, i.e.,

$$M_T := \{x \in \mathbb{X} : \|x\| = 1, \|Tx\| = \|T\|\}.$$

Birkhoff-James Orthogonality in $\mathbb{X}_{\boldsymbol{n}}^\infty$

Begin by observing that the dual of $\mathbb{X}_\infty^n$ is ${\mathbb{X}_1^*}^n$ with the functional $Psi_{(f_1, f_2, \dots, f_n)}$ corresponding to $(f_1, f_2, \dots, f_n) \in {\mathbb{X}_n^*}^1$ given by

$$\Psi_{(f_1, f_2, \dots, f_n)}(x_1, x_2, \dots, x_n) := \sum_{i=1}^{n} f_i(x_i), (x_1, x_2, \dots, x_n) \in \mathbb{X}_\infty^n.$$

We now characterize the support functionals of any non-zero element of $\mathbb{X}_\infty^n$.

Proposition 3. *Given* $x = (x_1, x_2, \dots, x_n) \in \mathbb{X}_\infty^n$ *non-zero,* $(f_1, f_2, \dots, f_n) \in {\mathbb{X}_1^*}^n$ *is a support functional of the element if and only if*

$$f_i = \lambda_i g_i, 1 \le i \le n, \qquad \lambda_i \in [0,1], \sum_{i=1}^{n} \lambda_i = 1,$$

where g_i *is a support functional of* x_i *for every* $1 \le i \le n$ *and* $\lambda_i = 0$ *if* $\|x_i\| < \|x\|$.

Proof: To prove the sufficiency, note that if $(f_1, f_2, \dots, f_n)$ satisfies the given condition, then

$$\|(f_1, f_2, \dots, f_n)\| = \sum_{i=1}^{n} \|f_i\| = \sum_{\mathrm{i}=1}^{\mathrm{n}} \lambda_i \|g_i\| = 1,$$

and

$$\Psi_{(f_1, f_2, \dots, f_n)}(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} \lambda_i g_i(x_i) = \sum_{\|x_i\| = \|x\|} \lambda_i g_i(x_i) = \sum_{\|x_i\| = \|x\|} \lambda_i \|x_i\| \\ = \|x\|,$$

since $\lambda_i = 0$ if $\|x_i\| \neq \|x\|$.

For the necessity, note that if $\Psi_{(f_1, f_2, \dots, f_n)}$ is a support functional of $(x_1, x_2, \dots, x_n)$, then

$$\|x\| = \sum_{i=1}^{n} f_i(x_i) \leq \sum_{i=1}^{n} \|f_i\| \|x_i\| \leq \sum_{i=1}^{n} \|f_i\| \|x\| = \|x\| \|(f_1, f_2, \dots, f_n)\|.$$

Since $\|(f_1, f_2, \dots, f_n)\| = 1$, equality must hold in all the inequalities giving $f_i = 0$ if $\|x_i\| < \|x\|$ and $f_i(x_i) = \|f_i\| \|x_i\|$ if $\|x_i\| = \|x\|$. Hence $f_i = \lambda_i g_i$ for some support functional g_i of x_i and $\lambda_i \geq 0$ if $\|x_i\| = \|x\|$. Finally,

$$\sum_{i=1}^{n} \lambda_i = \sum_{\|x_i\| = \|x\|} \lambda_i = \sum_{\|x_i\| = \|x\|} \lambda_i \|g_i\| = \sum_{\|x_i\| = \|x\|} \|f_i\| = \sum_{i=1}^{n} \|f_i\| = 1.$$

□

We are now ready to characterize Birkhoff-James orthogonality in $\mathbb{X}_\infty^n$.

Theorem 4. *Given two elements* $x = (x_1, x_2, \dots, x_n)$ *and* $y = (y_1, y_2, \dots, y_n)$ *of* $\mathbb{X}_\infty^n$, $x \perp_B y$ *if and only if either one of the following two conditions holds*

(1) $x_i \perp_B y_i$ *for some* $1 \leq i \leq n$ *such that* $\|x_i\| = \|x\|$.

(2) $[x_i, y_i]_1 [x_j, y_j]_2 < 0$ *for some* $1 \leq i < j \leq n$ *and* $[.,.]_1, [.,.]_2$ *two semi-inner products on* $\mathbb{X}$, *where* $\|x_i\| = \|x_j\| = \|x\|$.

Proof. We begin with the sufficiency. If condition 1 holds, then by the James' characterization of Birkhoff-James orthogonality, there exists a support functional f of x_i such that $f(y_i) = 0$. Define $\Gamma: \mathbb{X}_\infty^n \to \mathbb{R}$ given by

$$\Gamma\big((z_1, z_2, \dots, z_n)\big) \coloneqq f(z_i), (z_1, z_2, \dots, z_n) \in \mathbb{X}_\infty^n.$$

Then by Proposition 3, Γ is a support functional of x and $\Gamma(y) = 0$ giving $x \perp_B y$. If condition 2 holds, then define $f_i, f_j : \mathbb{X}_\infty^n \to \mathbb{R}$ given by

$$f_i(z) = \frac{1}{\|x_i\|}[x_i, z], f_j(z) = \frac{1}{\|x_j\|}[x_j, z], z \in \mathbb{X}.$$

Then f_i, f_j are support functionals of x_i and x_j respectively. Further, since $f_i(y_i)$ and $f_j(y_j)$ are of opposite signs, there exists $\lambda \in (0,1)$ such that $\lambda f_i(y_i) + (1-\lambda) f_j(y_j) = 0$. Define $\Gamma' : \mathbb{X}_\infty^n \to \mathbb{R}$ as

$$\Gamma'\big((z_1, z_2, \dots, z_n)\big) \coloneqq \lambda f_i(z_i) + (1-\lambda) f_j(z_j), (z_1, z_2, \dots, z_n) \in \mathbb{X}_\infty^n.$$

Then again by Proposition 3, Γ' is a support functional of x and $\Gamma'(y) = 0$ giving $x \perp_B y$.

For proving the necessity, consider $(f_1, f_2, \dots, f_n) \in \mathbb{X}_n^{*^1}$ such that $\Psi_{(f_1, f_2, \dots, f_n)}$ is a support functional of x that annihilates y. By Proposition 3, there exists $\lambda_i \in [0,1]$ and g_i support functionals of x_i such that

$$\sum_{\|x_i\| = \|x\|} \lambda_i g_i(y_i) = 0.$$

Therefore, $g_i(y_i) = 0$ for some $1 \le i \le n$, i.e., $x_i \perp_B y_i$ or $g_i(y_i)$ and $g_j(y_j)$ are of opposite signs for some $1 \le i < j \le n$ giving $g_i(y_i) g_j(y_j) < 0$. Finding semi-inner products $[.,.]_1$ and $[.,.]_2$ of $\mathbb{X}$ such that

$$[x_i, z] = \|x_i\| g_i(z) \; and \; [x_j, z] = \|x_j\| g_j(z), for\ every\ z \in \mathbb{X},$$

we get condition 2. □

This result allows us to characterize smoothness in this space.

Corollary 5. *A point $x = (x_1, x_2, \dots, x_n) \in \mathbb{X}_\infty^n$ is smooth if and only if there exists a unique $1 \le i \le n$ such that $\|x_i\| = \|x\|$ and x_i is smooth.*

Proof. For the necessity, note that if $\|x_i\| = \|x_j\| = \|x\|$ for some $1 \le i < j \le n$, then by Proposition 3, we can find more that one support functional of x. Also if $\|x_i\| = \|x\|$ and x_i is not smooth, then again by Proposition 3, we can find more than one support functional of x.

For the sufficiency, note that if the aforesaid condition holds, then by Theorem 4, $x \perp_B y$ for some $y = (y_1, y_2, \dots, y_n) \in \mathbb{X}_\infty^n$ if and only if $x_i \perp_B y_i$, where $\|x_i\| = \|x\|$. Hence, clearly, for $y = (y_1, y_2, \dots, y_n),\ z = (z_1, z_2, \dots, z_n) \in \mathbb{X}_\infty^n$,

$$x \perp_B y, x \perp_B z \Leftrightarrow x_i \perp_B y_i, x_i \perp_B z_i \Rightarrow x_i \perp_B (y_i + z_i) \Leftrightarrow x \perp_B (y + z).$$

Hence by the James' characterization of smoothness, x is smooth.

We now prove a necessary condition for an element of $\mathbb{X}_\infty^n$ to be right-symmetric.

Theorem 6. *If* $x = (x_1, x_2, \ldots, x_n) \in \mathbb{X}_\infty^n$ *is a right-symmetric point of* $\mathbb{X}_\infty^n$*, then* $\|x_i\| = \|x\|$ *for every* $1 \leq i \leq n$.

Proof. For the sake of contradiction, let x be a right-symmetric point of $\mathbb{X}_\infty^n$ and without loss of generality, assume that $\|x_n\| < \|x\|$. Consider $y = (y_1, y_2, \ldots, y_n) \in \mathbb{X}_\infty^n$ such that $\|y_i\| < \|y\|$ for every $1 \leq i \leq n$ and $y_n \perp_B x_n$. Now observe that for any $z \in \mathbb{X}$ non-zero,

$$J(z) \coloneqq \{f \in \mathbb{X}^*: \|f\| = 1, f(z) = \|z\|\},$$

is a convex set and therefore is connected. Further $J(z)$ is a closed subset of the closed unit ball of $\mathbb{X}^*$ under the weak* topology and hence is weak* compact by the Banach-Alaoglu theorem [15]. Therefore,

$$I(w) \coloneqq \{f(w): f \in J(z)\}, w \in \mathbb{X},$$

is a compact connected subset of $\mathbb{R}$ and therefore is a finite closed interval.

Hence if z is not Birkhoff-James orthogonal to w, then $I(w)$ is contained either in the positive real line or in the negative real line, i.e., $[z, w]$ does not change sign as $[.,.]$ varies over all the semi-inner products on $\mathbb{X}$.

Therefore, we can choose $y_i \in \mathbb{X}$ such that $[x_i, y_i] > 0$ for every $1 \leq i < n$ and every semi-inner product $[.,.]$ on $\mathbb{X}$. For this y, we get by Theorem 4, $y \perp_B x$ but x is not Birkhoff-James orthogonal to y. □

The following remark will be required later.

Remark 7. *If* $x = (x_1, x_2, \ldots, x_n) \in \mathbb{X}_\infty^n$ *is such that* $\|x_i\| < \|x\|$*, then given any* $\epsilon > 0$*, we can find* $y = (y_1, y_2, \ldots, y_n) \in \mathbb{X}_\infty^n$ *satisfying* $y \perp_B x$ *but* x *is not Birkhoff-James orthogonal to* y *such that* $\|y_i\| = \|y\|$ *and* $\|y_j\| < \epsilon$ *for every* $j \neq i$.

Point-wise Symmetry of Birkhoff-James Orthogonality and the Geometry of $\mathbb{B}(\ell_\infty^n, \ell_1^m)$

In this section, we show that any non-zero left-symmetric point of $\mathbb{B}(\ell_\infty^n, \ell_1^m)$ is smooth and any unit norm right-symmetric point of the space is an extreme point of the closed unit ball. We begin by characterizing the smooth points of the space of operators between two finite-dimensional Banach spaces.

Theorem 8. *Given finite-dimensional Banach spaces* $\mathbb{X}$ *and* $\mathbb{Y}$*, an operator* $T \in \mathbb{B}(\mathbb{X}, \mathbb{Y})$ *is smooth if and only if* $M_T = \{x_0, -x_0\}$ *for some* $x_0 \in \mathbb{X}$ *and* Tx_0 *is a smooth point of* $\mathbb{Y}$.

Proof. For proving the sufficiency, note that if $M_T = \{x_0, -x_0\}$, then by [16, Corollary 2.2.1], $T \perp_B T'$ for some $T' \in \mathbb{B}(\mathbb{X}, \mathbb{Y})$ if and only if $Tx_0 \perp_B T'x_0$. Hence by smoothness of Tx_0 we get

$$\begin{aligned} T \perp_B S, T \perp_B S' \Leftrightarrow Tx_0 \perp_B Sx_0, Tx_0 \perp_B S'x_0 \Rightarrow Tx \perp_B (Sx_0 + S'x_0) \\ \Leftrightarrow T \perp_B (S + S'). \end{aligned}$$

Thus by the James' characterization of smoothness, T is smooth.

For proving the necessity, note that if $x \in M_T$ and $Tx \in \mathbb{Y}$ is not smooth, then there exist f and g distinct support functionals of Tx giving two distinct support functionals Φ and Ψ of T given by

$$\Phi(S) \coloneqq f(Sx), \Psi(S) \coloneqq g(Sx), S \in \mathbb{B}(\mathbb{X}, \mathbb{Y}).$$

Also if M_T contains two linearly independent points x and y, then consider support functionals f and g (not necessarily distinct) of Tx and Ty respectively to get two distinct support functionals Φ and Ψ of T given by

$$\Phi(S) \coloneqq f(Sx), \Psi(S) \coloneqq g(Sy), S \in \mathbb{B}(\mathbb{X}, \mathbb{Y}).$$

□

We now prove the relationship between left-symmetric points and smoothness in $\mathbb{B}(\ell_\infty^n, \ell_1^m)$.

Theorem 9. *A non-zero element* $T \in \mathbb{B}(\ell_\infty^n, \ell_1^m)$ *is smooth if it is left-symmetric.*

Proof. Let $x \in M_T$ and assume that Tx is not a left-symmetric point of ℓ_1^m. Then find $y \in \ell_1^m$ such that $Tx \perp_B y$ but y is not Birkhoff-James orthogonal to x. Construct $T' \in \mathbb{B}(\ell_\infty^n, \ell_1^m)$ such that $M_{T'} = \{x, -x\}$ and $T'x = y$. Hence by [16, Corollary 2.2.1], we get that $T \perp_B T'$ but T' is not Birkhoff-James orthogonal to T. Thus Tx must be a left-symmetric point of ℓ_1^m whenever $x \in M_T$. However, from [6], we get that every left-symmetric point of ℓ_1^m is smooth. Hence Tx must be a smooth point of ℓ_1^m for every $x \in M_T$.

Now, if M_T contains more than two points, then M_T must contain at least two extreme points. Since $x \perp_B y$ for any two linearly independent extreme points of ℓ_∞^n, we can find rank 1 operator $T' \in \mathbb{B}(\ell_\infty^n, \ell_1^m)$ such that $M_{T'} = \{x, -x\}$, $T'x = Tx$ and $T'y = 0$ for every extreme point $y \in M_T$ linearly independent to x. Hence again, $T \perp_B T'$ but T' is not orthogonal to T.

Thus $M_T = \{x, -x\}$ and Tx is a smooth point of ℓ_1^m. Hence by Theorem 8, T must be smooth. □

We now come to our final result, where we relate the notion of right-symmetry of Birkhoff-James orthogonality with extreme contractions. Recall from [8] that these are simply defined to be the extreme points of the closed unit ball of $\mathbb{B}(\ell_\infty^n, \ell_1^m)$.

Proof of Theorem 1. Let us denote ℓ_∞^n and ℓ_1^m by $\mathbb{X}$ and $\mathbb{Y}$ respectively. Begin by observing that if M_T contains n linearly independent extreme points of ℓ_∞^n, then T is an extreme contraction since if $T = \frac{1}{2}(T_1 + T_2)$, for some $T_1, T_2 \in \mathbb{B}(\ell_\infty^n, \ell_1^m)$ having norm 1, then $Tx = \frac{1}{2}(T_1 x + T_2 x)$ for every extreme point x of ℓ_∞^n contained in M_T, giving $T_1 x = T_2 x = Tx$. Since there are n-many linearly independent x in M_T, clearly T, T_1 and T_2 agree on ℓ_∞^n, proving T to be an extreme contraction.

Now, assume the contrary. As T is not an extreme contraction, T cannot attain norm at more than $n - 1$ linearly independent extreme points. Let T attain norm at $x_1, x_2, \dots, x_k$, which are linearly independent extreme points of $\mathbb{X}$. Extend $\{x_1, x_2, \dots, x_k\}$ to a basis $\{x_1, x_2, \dots, x_n\}$ of $\mathbb{X}$ consisting of extreme points. Also, by composing T with a suitable signed permutation operator, we can assume that $x_i = \sum_{j=1}^n e_j - 2e_i$, where e_i denotes the i-th standard basis vector.

Define a map $\Gamma: \mathbb{B}(\mathbb{X}, \mathbb{Y}) \to \mathbb{Y}_\infty^n$ given by

$$\Gamma(S) \coloneqq (Sx_1, Sx_2, \dots, Sx_n), S \in \mathbb{B}(\mathbb{X}, \mathbb{Y}).$$

Clearly Γ is a bijective bounded linear map having norm 1 and it attains norm at T. Now let $T \perp_B S$ for some $S \in \mathbb{B}(\mathbb{X}, \mathbb{Y})$. Then by [16, Theorem 2.2], there exist $x, x' \in M_T$ such that

$$\|Tx + \lambda Sx\| \geq \|Tx\|, \|Tx' - \lambda Sx'\| \geq \|Tx'\| \ \ for\ \ every\, \lambda \geq 0.$$

Now if $\|Tx + \lambda Sx\| \geq \|Tx\|$ for every $\lambda > 0$ then either $Tx \perp_B Sx$ or $[Tx, T'x] > 0$ for evert semi-inner product $[.,.]$ on $\mathbb{Y}$ since otherwise there exists $\delta > 0$ such that $f(T'x) \leq -\delta$ for every support functional f of Tx. However, this means that for any support functional f of Tx,

$$\|Tx\| = f(Tx) \geq f(Tx + \lambda Sx) - \delta\lambda.$$

Since the support functionals of Tx are limit points of the set of support functionals of $Tx + \lambda Sx$ for every $\lambda > 0$, this violates $\|Tx + \lambda Sx\| \geq \|Tx\|$ for every $\lambda \geq 0$.

Now if $x \in M_T$, then x lies on the convex hull of $\{x_1, x_2, \dots, x_k\}$. If A is the smallest subset of $\{x_1, x_2, \dots, x_k\}$ such that $x \in conv(A)$, then $Tx = Tx_i$ for every $x_i \in A$. Hence if $Tx \perp_B Sx$ and $x = \sum_{i=1}^n \lambda_i x_i, \lambda_i > 0, \sum_{i=1}^n \lambda_i = 1$, then find a support functional f of Tx such that $f(Sx) = 0$. Since f is a support functional of Tx_i for every $x_i \in A$,

$$\Psi: y \mapsto \sum_{x_i \in A} \lambda_i f(y_i)\,, y = (y_1, y_2, \dots, y_n) \in \mathbb{Y}_\infty^n,$$

is a support functional of $\Gamma(T)$ by Proposition 3, such that $\Psi\big(\Gamma(S)\big) = 0$. Also, if $[Tx, Sx] > 0$, then $[Tx_i, Sx] > 0$ for every $x_i \in A$. Hence

$$\sum_{x_i \in A} \lambda_i [Tx_i, Sx_i] > 0,$$

giving $[Tx_i, Sx_i] > 0$ for some $x_i \in A$. Similarly if $[Tx, Sx] < 0$, then $[Tx_i, Sx_i] < 0$ for some $x_i \in A$. Now as $T \perp_B S$, $Tx \perp_B Sx$ for some $x \in M_T$ or $[Tx, Sx] < 0$ and $[Ty, Sy] > 0$ for some $x, y \in M_T$. In either case, by Theorem 4, $\Gamma(T) \perp_B \Gamma(S)$.

Now, by Theorem 6, $\Gamma(T)$ is not a right-symmetric points of $\mathbb{Y}_\infty^n$. Further, by Remark 7, we can find $y = (y_1, y_2, \dots, y_n) \in \mathbb{Y}_\infty^n$ such that $\Gamma(T)$ is not Birkhoff-James orthogonal to y and $y \perp_B \Gamma(T)$, with $\|y_n\| = \|y\|$ and $\|y_i\|$ arbitrarily small for $i \neq n$. Now let x be any extreme point of $\mathbb{X}$. Then there exists a partition (A, B) of $\{1,2, \dots, n\}$ such that

$$x = \sum_{i \in A} e_i - \sum_{i \in B} e_i.$$

Then if $A = \emptyset$ or $B = \emptyset$,

$$Sx = \frac{1}{(n-2)} \sum_{i=1}^{n} Sx_i\,, for\ every\ S \in \mathbb{B}(\mathbb{X}, \mathbb{Y}).$$

Else, if $|A|, |B| \geq 2$, then

$$Sx = \frac{|A| - |B|}{2(n-2)} \sum_{i=1}^{n} Sx_i + \frac{1}{2}\left(\sum_{i \in A} Sx_i - \sum_{i \in B} Sx_i\right), for\ every\ S \in \mathbb{B}(\mathbb{X}, \mathbb{Y}).$$

Since $n \geq 4$, by choosing $\|y_i\|$ sufficiently small, we can ensure that $\Gamma^{-1}(y)$ attains norm only at $\{e_n, -e_n\}$.

Therefore, by [16, Corollary 2.2.1], $\Gamma^{-1}(y) \perp_B T$. However, as $\Gamma(T)$ is not Birkhoff-James orthogonal to y, we must have T not Birkhoff-James orthogonal to $\Gamma^{-1}(y)$, violating the right-symmetry of T. □

Acknowledgments

The research of Dr. Babhrubahan Bose is funded by PMRF research fellowship under the supervision of Prof. Apoorva Khare and Prof. Gadadhar Misra.

References

[1] B. Bagchi, G. Misra, *"On Grothendieck Constants",* preprint, 2008.

[2] G. Birkhoff, *"Orthogonality in linear spaces",* Duke Math J., 1 (1935), 169-172.

[3] R. Blei *"The Grothendieck inequality revisited",* Mem. Amer. Math. Soc., (2014).

[4] B. Bose, S. Roy, D. Sain, *"Birkhoff-James orthogonality and its local symmetry in some sequence spaces",* Revista de la Real Academia de Ciencias Exactas, Fiscias y Naturales. Serie A. Mathemáticas 117.3 (2023) 93.

[5] B. Bose, *"Birkhoff-James orthogonality and its point-wise symmetry in some function spaces",* arxiv:2205.13078 [math.FA].

[6] A. Chattopadhayay, D. Sain, T. Senapati, *"Characterization of symmetric points in* l_p^n*-spaces",* Lin. Mult. Alg., 69.16 (2021), 2998-3009.

[7] A. Grothendieck, *"Resumé de la théorie métrique des produits tensoriels topologiques",* Bull. Soc. Mat. Sao-Paolo, 8 (1953) 1-79, reprinted in Resenhas 2.4 (1996), 401-480.

[8] A. Iwanik, *"Extreme contractions on certain function spaces",* Coll. Math., 40.1 (1978), 147-153.

[9] R. C. James, *"Inner product in normed linear spaces",* Bull. Amer. Math. Soc., 53 (1947), 559-566.

[10] R. C. James, *"Orthogonality and linear functionals in normed linear spaces",* Trans. Amer. Math. Soc., 61 (1947), 265-292.

[11] S. Khot, A. Naor, *"Grothendieck-type inequalities in combinatorial optimization",* Comm. Pure Appl. Math., 65.7 (2012), 992-1035.

[12] J. L. Krivine, *"Constantes de Grothendieck et fonctions de type positif sur les spheres",* Adv. in Math., 31 (1979), 16-30.

[13] A. Lima, *"The geometric structure of* $L(\ell_\infty^3, \ell_1^3)$*",* preprint, 1979.

[14] G. Pisier, *"Grothendieck's theorem, past and present",* Bull. Amer. Math. Soc., 49 (2012), 237-323.

[15] W. Rudin, *"Functional Analysis",* Mathematics Series, McGraw-Hill, (1991).

[16] D. Sain, *"Birkhoff-James orthogonality of linear operators on finite-dimensional Banach spaces",* J. Math. Anal. Appl., 447.2 (2017), 860-866.

[17] D. Sain, *"On the norm attainment set of a bounded linear operatot",* J. Math. Anal. Appl., 457.1 (2018), 67-76.