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In 1969, Jean-Marie Souriau introduced "Lie Groups Thermodynamics" within the framework of Geometric Mechanics, offering a novel approach to Statistical Mechanics. F. Barbaresco and collaborators have demonstrated the applicability of Souriau's model across various domains, including information geometry and geometric deep learning. This paper presents a comprehensive review of the extension of Souriau's symplectic model to Quantum Information Theory. Building on the work of F. Barbaresco and F. Guy-Balmaz, who highlighted strong analogies between quantum information geometry and Lie group thermodynamics, this review explores the role of unitary representations of Lie algebras and the equivalence between the Fisher metric and the Bogoliubov-Kubo-Mori metric. In addition to the review, this paper introduces new results that further extend the classical Souriau framework by integrating modern developments in quantum thermodynamics. Specifically, this work links "Quantum Lie Groups Thermodynamics" to the geometry of coadjoint orbits, utilizing a geometric framework for mixed quantum states based on a Kähler structure. This framework, which encompasses a symplectic form, an almost complex structure, and a Riemannian metric, offers a comprehensive characterization of the space of mixed quantum states, providing deeper insights into the underlying geometric structure of quantum thermodynamics.

Introduction

J. M. Souriau developed a geometric theory of statistical mechanics, known as Lie Groups Thermodynamics, to address the issue that Gibbs equilibrium states do not conform to standard physical covariance principles. This theory is grounded in the Hamiltonian action of a Lie group on a symplectic manifold, where generalized Gibbs states are associated with a parameter β from the Lie algebra, which acts as a geometric analog to temperature (akin to Planck's temperature). In this framework, conventional Gibbs states, derived from a Hamiltonian, emerge as particular instances where the Lie group simplifies to a one-parameter group. For much of the material in this section I have been guided by the contents of (Barbaresco & Gay-Balmaz, 2020; Marle, 2020; Souriau, 1969).

Pioneering the Microscopic Theory of Matter

The origins of statistical mechanics can be traced back to Daniel Bernoulli (1700–1782), who, in his seminal work *Hydrodynamica* published in 1738, proposed a revolutionary idea: he conceived fluids (both gases and liquids) as composed of a vast number of moving particles. Bernoulli explained that the pressure exerted by a fluid results from the collisions of these particles against the walls of the container or any surface that measures the pressure. This concept laid the groundwork for understanding the microscopic basis of macroscopic phenomena, although it was not immediately recognized or widely accepted by the scientific community of his time.

Bernoulli's insight into the particle nature of fluids remained largely overlooked for more than a century. It wasn't until the mid-19th century that his ideas gained traction, thanks to the work of several pioneering scientists. Rudolf Clausius (1822–1888) was among the first to build on Bernoulli's concept, beginning in 1857 to develop what became known as the kinetic theory of gases. Clausius aimed to explain macroscopic properties of gases, such as temperature and pressure, by deriving them from the equations governing the motion of individual particles.

Around 1860, James Clerk Maxwell (1831–1879) made a significant advancement by determining the probability distribution of particle velocities in a gas at thermodynamic equilibrium, now known as the Maxwell-Boltzmann distribution. Ludwig Boltzmann (1844–1906) extended this work further. In 1872, he derived an evolution equation for the probability distribution of particle velocities in gases that are not in equilibrium, now famously known as the Boltzmann equation. By employing probabilistic arguments to describe particle collisions, Boltzmann introduced a quantity denoted by H, which he showed always decreases over time—a concept now recognized as a precursor to the second law of thermodynamics. Boltzmann's H-theorem effectively relates to the concept of entropy, with the H function being the negative of what we now understand as entropy.

Building on these developments, Josiah Willard Gibbs (1839–1903) formalized the principles of what would become a new branch of theoretical physics—

statistical mechanics. Gibbs' framework provided the tools to link microscopic behavior with macroscopic observables, offering a profound understanding of thermodynamics from a statistical perspective.

In the first half of the 20th century, it became clear that the motion of molecules does not fully adhere to the laws of classical mechanics as initially assumed. Instead, quantum mechanics was found to govern these microscopic systems. However, the foundational concepts of statistical mechanics established by Gibbs proved to be general and robust enough to be adapted to the quantum mechanical framework. This adaptability allowed statistical mechanics to be applied not only to gases but also to liquids and solids, thus broadening its scope and cementing its place as a cornerstone of modern physics.

Symplectic Souriau Lie Group Thermodynamics Model

In this section we will introduce Souriau's Lie Group Thermodynamic model in the symplectic case.

Notations and Definitions

For our setting we will need:

- A symplectic manifold (M, ω) is a manifold M endowed with a closed nondegenerate valued 2-form where $\dim(M) = 2n$
- A symplectic Lie group action $\phi: G \times M \to M$ of G on M is symplectic, meaning
$$\phi_g^* \omega = \omega, \quad \forall g \in G.$$
- The action is Hamiltonian and admits a symplectic momentum map
$$\mathfrak{J}: M \to \mathfrak{g}^*,$$
which satisfies
$$i_{\xi_M} \omega = d\mathfrak{J}_\xi,$$
where $\mathfrak{J}_\xi: M \to \mathbb{R}$ is defined by
$$\mathfrak{J}_\xi(m) = \mathfrak{J}(m).\xi = \langle J(m), \xi \rangle$$
and ξ_M is the infinitesimal generator of the action corresponding to $\xi \in \mathfrak{g}$ the Lie algebra of G and $\mathfrak{g}^*$ the dual algebra.
- if M is connected, there is a group one-cocycle $\theta \in C^\infty\big(G, L(\mathfrak{g}, \mathbb{R}^n)\big), \theta$ defined by
$$\theta(g) = \mathfrak{J}\big(\Phi(m)\big) - Ad_{g^{-1}}^*\big(\mathfrak{J}(m)\big)$$
where $Ad_{g^{-1}}^*$ is the coadjoint representation action.
- $\mathfrak{g}$ the Lie algebra of G, whose elements will be denoted β the generalisations of the inverse temperature.
- A duality pairing between elements ν of the dual space $\mathfrak{g}^*$ and elements $\beta \in \mathfrak{g}$ denoted as $\langle \nu, \beta \rangle$.
- The manifold M is endowed with the Liouville volume form $d\mu$.

Denote by $\Omega \subset \mathfrak{g}$ the largest open set such that for all $\beta \in \Omega$ the two integrals

$$\int_M e^{-\langle \Im(m),\beta \rangle} d\mu \in \mathbb{R} \text{ and } \int_M \Im(m) e^{-\langle \Im(m),\beta \rangle} d\mu \in \mathfrak{g}^* \tag{1}$$

converge.

On Ω one can define the so-called partition function (or characteristic function) $\psi: \Omega \to \mathbb{R}$, given by

$$\psi(\beta) = \int_M e^{-\langle \Im(m),\beta \rangle} d\mu. \tag{2}$$

This permits us to define the generalized Gibbs probability densities associated to each $\beta \in \Omega$:

$$p_\beta(m) = \frac{1}{\psi(\beta)} e^{-\langle \Im(m),\beta \rangle} \tag{3}$$

Remark 1. *There are two facts which are very important to mention:*

- *For applications in information geometry, it is required that $\beta \to p_\beta$ is injective.*
- *The Gibbs densities are not defined on the whole vector space $\mathfrak{g}$ but only on the open subset Ω. From now on we assume that Ω is not empty. Its elements are called geometric temperatures.*
- *If the integrals defining Ω are normally convergent then for example Ω is convex.*

The generalized probability density ψ can be written using the Massieu potential $\Phi: \Omega \to \mathbb{R}$

$$p_\beta(m) = e^{\Phi(\beta) - \langle \Im,\beta \rangle}, \quad \forall \beta \in \Omega \tag{4}$$

and

$$\Phi(\beta) = -\log(\psi(\beta)) \tag{5}$$

Another interesting quantity in this framework is the thermodynamic heat $Q: \Omega \to \mathfrak{g}^*$ is the first derivative of the Massieu potential, i.e.,

$$Q(\beta) := D\Phi(\beta) = \int_M \Im(m) p_\beta(m) d\mu = \mathbb{E}_\beta[\Im] \in \mathfrak{g}^* \tag{6}$$

where $\mathbb{E}_\beta$ denotes the expectation with respect to p_β.

Entropy and Some Associated Results

Let Ω^* be the image of the Ω by the thermodynamic heat Q and assume that $Q: \Omega \to \Omega^*$ is a diffeomorphism. In this case, the entropy $s: \Omega^* \to \mathbb{R}$ can be defined as as the Legendre transform of the Massieu potential $\Phi: \Omega \to \mathbb{R}$, namely

$$s(\nu) := \langle \nu, \beta \rangle - \Phi(\beta), \text{ where } \beta = Q^{-1}(\nu) \quad (7)$$

The next result has been proved in (Barbaresco & Gay-Balmaz, 2020).

Lemma 2. *For every $\beta \in \Omega$, the following equality holds true:*

$$s\big(Q(\beta)\big) = S\big(p_\beta\big), \text{ where } S(p) = -\int_M p \log p d\mu. \quad (8)$$

S is the entropy of the probability density p and $Q(\beta)$ is the thermodynamic heat.

The following equivariance properties have been proved in a general framework in (Barbaresco & Gay-Balmaz, 2020) and applied there also for the symplectic case:

$$(Ad_g)\Omega = \Omega, \quad \psi\Big((Ad_g)\beta\Big) = \psi(\beta) e^{\langle \theta(g^{-1}), \beta \rangle}, \quad p_\beta \circ \phi_g = p_{\left(Ad_{g^{-1}}\right)\beta}$$

and

$$\Phi\Big((Ad_g)\beta\Big) = \Phi(\beta) - \langle \theta(g^{-1}), \beta \rangle \quad (9)$$

$$Q\Big((Ad_g)\beta\Big) = \Big(Ad^*_{g^{-1}}\Big)\big(Q(\beta)\big) + \theta(g) \quad (10)$$

$$s\Big(\Big(Ad^*_{g^{-1}}\Big)(\nu) + \theta(g)\Big) = s(\nu) \quad (11)$$

for every $g \in G$.

Furthermore Ω^* is invariant under the affine action $\nu \in \mathfrak{g}^* \mapsto \Big(Ad^*_{g^{-1}}\Big)(\nu) + \theta(g) \in \mathfrak{g}^*$

-
- *Fisher Metric and the maximum entropy principle*

The generalized heat capacity is the symmetric tensor field $K: \Omega \to sym(\mathfrak{g})$, defined as minus the Hessian matrix of the Massieu potential, namely

$$K(\beta) := -D^2 \Phi(\beta) = D^2 \log \psi(\beta) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}. \quad (12)$$

As proved in (Barbaresco & Gay-Balmaz, 2020) the generalized heat capacity has the following expression:

$$\begin{aligned} K(\beta) &= \mathbb{E}_\beta\big[\big(\mathfrak{J} - \mathbb{E}_\beta[\mathfrak{J}]\big) \otimes \big(\mathfrak{J} - \mathbb{E}_\beta[\mathfrak{J}]\big)\big] \\ &= \mathbb{E}_\beta\big[\big(\mathfrak{J} - Q(\beta)\big) \otimes \big(\mathfrak{J} - Q(\beta)\big)\big] \end{aligned}$$

As a consequence, $K(\beta)$ is positive semidefinite for all $\beta \in \Omega$. Being the derivative of $Q: \Omega \to \Omega^*$ it is positive definite if Q is a diffeomorphism. As noticed by F. Barbaresco in many of his papers there is a link between Souriau's work and the information geometry that the authors apply in this general framework. Recall that in information geometry, the Fisher metric associated with the family p_β, $\beta \in \Omega$ of probability densities is the symmetric tensor field $I: \Omega \to sym(E)$ defined by

$$I(\beta) = -\mathbb{E}_\beta\left[D^2 \log p_\beta\right]. \qquad (13)$$

The authors in (Barbaresco & Gay-Balmaz, 2020) proved that in fact the generalized heat capacity of p_β and the Fisher metric of p_β are equal:

$$I(\beta) = K(\beta) = D^2 \log\psi(\beta)$$

Furthermore they were able to show that if we assume that $Q: \Omega \to \Omega^*$ is a diffeomorphim then the inverse of the Fisher metric, i.e., the cometric on Ω^* induced from the Fisher metric on Ω, is given by minus the Hessian of the entropy:

$$-D^2 s(\nu) =: E^* \times E^* \to \mathbb{R}, \quad \forall \nu \in \Omega^*.$$

There is also an equivariance property satisfied by the Fisher metric namely:

$$K\left(\left(Ad_g\right)\beta\right)\left(\left(Ad_g\right)(\delta\beta_1), \left(Ad_g\right)(\delta\beta_2)\right) = K(\beta)(\delta\beta_1, \delta\beta_2),\ \forall g \in G. \qquad (14)$$

One of the important results proved in the previous cited paper shows that the generalized Gibbs probability densities satisfy the maximum entropy principle.

Proposition 3. *(Maximum entropy principle). Let* $\mathfrak{J}: M \to \mathfrak{g}^*$ *be a smooth function and* $\nu \in \Omega^*$ *a given element. The generalized Gibbs probability density* p_β *in Equation ([gibbs_density1]) with* $\beta = Q^{-1}(\nu)$ *is a solution of the maximum entropy principle:* $\max_q\left[-\int_M q \log(q) d\mu\right]$ such that $\begin{cases} \int_M q\, d\mu & = 1 \\ \int_M \mathfrak{J}\, q d\mu & = \nu \end{cases}$

Quantum Souriau Lie Group Thermodynamics Model

In this section we apply the general framework (Barbaresco & Gay-Balmaz, 2020) for Gibbs probability densities in statistical mechanics and information geometry in the case of Unitary Representations and Quantum Fisher Metric as studied in the above cited paper.

- *Notations and definitions*

For our setting we will need:

- Let G be a Lie group acting on a complex Hilbert space $\mathcal{H}$ by a unitary left representation,

$$\mathcal{U}_g: \mathcal{H} \to \mathcal{H}$$

- Let $\beta_{\mathcal{H}}$ be the associated infinitesimal generator, giving the Lie algebra representation and which is defined in the standard way by

$$\beta_{\mathcal{H}}(m) = \frac{d}{dt}\mathcal{U}\big(\exp(t\beta)\big)(m)|_{t=0} \quad \text{for } m \in \mathcal{H}, \tag{15}$$

and consider the auto-adjoint operator

$$\imath\beta_{\mathcal{H}}.$$

We assume $\dim\mathcal{H} < \infty$. The following class of density matrices is considered

$$\rho_\beta = \frac{1}{\psi(\beta)}\exp(-\imath\beta_{\mathcal{H}}) \tag{16}$$

for $\beta \in \mathfrak{g}$, with partition function

$$\psi(\beta) = \mathrm{Tr}\big(\exp(-\imath\beta_{\mathcal{H}})\big) \tag{17}$$

As remarkerd in (Barbaresco & Gay-Balmaz, 2020) the expression of the density matrices in ([density_matrices]) represent a generalization for the class of density matrices, which includes the class considered in (Nencka & Streater, 1999) and references therein.

Unitary Representation of Lie Groups

In the previous section we saw that F. barbaresco and F. Guy-Balmaz introduced the auto-adjoint operator $-\imath\beta_{\mathcal{H}}$ in the density matrix ρ_β as a generalization of the unitary representations for the Lie algebras $\mathfrak{so}(3)$ and $\mathfrak{sl}(2,\mathbb{R})$ studied in (Nencka & Streater, 1999). There was no mathematical justification for this choice which we will provide using arguments from the book of Kirillov (Kirillov, 2004) in the appendix V. titled representation theory.

<u>The Space $\mathcal{H}^\infty$ of Smooth Vectors</u>

If $\mathcal{U}$ is a unitary representation of Lie group G on $\mathcal{H}$ then the matrix elements of $(\mathcal{U},\mathcal{H})$ are bounded and continuous but not necessarily smooth functions especially if we are considering the infinite dimensional case.

Definition 4. *A vector $m \in \mathcal{H}$ is called smooth if the vector-function $g \mapsto \mathcal{U}(g)m$ from G to $\mathcal{H}$ is strongly infinitely differentiable. The set of all smooth vectors in $\mathcal{H}$ is denoted by $\mathcal{H}^\infty$.*

As it was remarked by Kirillov in his book, the space $\mathcal{H}^\infty$ of smooth vectors is always dense in $\mathcal{H}$ and it is big enough to reconstruct the representation $(\mathcal{U},\mathcal{H})$ of a connected Lie group G from the representation of the Lie algebra $\mathfrak{g}$ in $\mathcal{H}^\infty$.

<u>Self-adjoint Infinitesimal Generators of Unitary Presentations with Domain $\mathcal{H}^\infty$</u>

The following theorem from (Kirillov, 2004) proves the existence of auto-adjoint infinitesimal generators of unitary presentations with domain $\mathcal{H}^\infty$ and thus justifying the generalization in the work of Barbaresco et al. in (Barbaresco & Gay-Balmaz, 2020):

Theorem 5. (Kirillov, 2004) *Let* $(\mathcal{U}, \mathcal{H})$ *be a unitary representation of Lie group* G. *Then:*

1. *The subspace* $\mathcal{H}^{\infty}$ *of smooth vectors is dense in* $\mathcal{H}$ *and stable with respect to all operators* $\mathcal{U}(g)$, $g \in G$.
2. *For any* $\beta \in \mathfrak{g}$, *the Lie algebra of* G, *the operator*
$$A = -\imath\mathcal{U}_*(\beta) =: -\imath\frac{d}{dt}\mathcal{U}\big(\exp(t\beta)\big)|_{t=0}, \quad (18)$$
with the domain $\mathcal{H}^{\infty}$ *is essentially auto-adjoint.*
3. *If* G *is connected, the representation* $(\mathcal{U}, \mathcal{H})$ *is completely determined by the representation* $\mathcal{U}_*$ *of* $\mathfrak{g}$ *in* $\mathcal{H}^{\infty}$ *defined by (*18*). In particular, for any* $m \in \mathcal{H}^{\infty}$ *we have*
$$\mathcal{U}\big(\exp(t\beta)\big)m = \big(\exp(\imath tA)\big)m \quad (19)$$
where the right-hand side is defined as the solution to the ordinary differential equation
$$m'(t) = \imath Am(t) \quad (20)$$
with the initial condition $m(0) = m$.

As said earlier the proof of this theorem is from the book of Kirillov in the appendix V. about representation theory Theorem 4. In the following we will collect the preparatory steps for its demonstration as they are highly informative.

<u>A summary of the different steps of Kirillov's theorem proof</u>

- For any $\beta \in \mathfrak{g}$ there correspond two vector fields on G: the left-invariant field R_β (infinitesimal right shift) and the right-invariant field L_β (infinitesimal left shift) given by
$$\big(R_\beta f\big)(g) := \frac{d}{dt}f\big(g\exp(t\beta)\big)\Big|_{t=0}, \quad \big(L_\beta f\big)(g) := \frac{d}{dt}f\big(\exp(-t\beta)\big)g\Big|_{t=0}, \quad (21)$$
- For any function $\phi \in \mathcal{A}(G)$ define the operator $\mathcal{U}(\phi)$ on $\mathcal{H}$ by the expression
$$\mathcal{U}(\phi) = \int_G \phi\,(g)\mathcal{U}(g)d_l g \quad (22)$$
where $d_l g$ is a left-invariant measure on G.
- The following relation holds:
$$\mathcal{U}_*(\beta)\mathcal{U}(\phi) = \mathcal{U}\big(L_\beta\phi\big) \quad (23)$$
- On $\mathcal{H}^{\infty}$ the following relation holds:
$$\mathcal{U}(\phi)\mathcal{U}_*(\beta) = -\mathcal{U}\left(\left(R_\beta + \mathrm{tr}(ad\beta)\right)\phi\right). \quad (24)$$
- From (23) it follows that for any $\phi \in \mathcal{A}(G)$ and any $m \in \mathcal{H}$ the vector $\mathcal{U}(\phi)m$ is smooth and therefore $\mathcal{U}(\phi)$ is called a smoothing operator. Kirillov then proves that the linear span of the images of all smoothing operators $\mathcal{U}(\phi)$, $\phi \in \mathcal{A}(G)$, is dense in $\mathcal{H}$.

- For $\beta \in \mathfrak{g}$ consider the operator $A = -\imath\mathcal{U}_*(\beta) = -\imath\frac{d}{dt}\mathcal{U}(\exp(t\beta))|_{t=0}$ with $\mathcal{H}^\infty$ as the domain of definition. Unitarity of $\mathcal{U}(g)$ leads to the fact that A is symmetric: $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x, y \in \mathcal{H}^\infty$. To show that a symmetric operator A with domain D_A is essentially auto-adjoint, one uses the equivalence in the following criterion from Appendix IV.2.3 in (Kirillov, 2004) :
$$\mathrm{Ker}(A^* + \imath \cdot \mathrm{Id}) = \mathrm{Ker}(A^* - \imath \cdot \mathrm{Id}) = \{0\}.$$
- The relation (19) follows from the very definition of $\exp(\imath tA)$.
- To reconstruct $(\mathcal{U}, \mathcal{H})$ from $(\mathcal{U}_*, \mathcal{H}^\infty)$ Kirillov argues in the following way. For any auto-adjoint operator $\bar{A}$ the operator $\exp(\imath t\bar{A})$ can be defined as follows. In the appropriate realization of $\mathcal{H}$ in the form $L^2(\beta, \mu)$ the operator $\bar{A}$ is just the multiplication by a real-valued function $a(x)$. Then one defines $\exp(\imath t\bar{A})$ as a multiplication by $\exp(\imath ta(x))$.
- When $\bar{A}$ is the closure of A, this definition coincides with (18) on the subspace $D_A = \mathcal{H}^\infty$. Therefore $\mathcal{U}(\exp(t\beta))$ coincides with $\exp(\imath tA)$ on $\mathcal{H}^\infty$ and with $\exp(\imath t\bar{A})$ on the whole space $\mathcal{H}$. Therefore, we know that the operators $\mathcal{U}(g)$ are in the neighborhood of the unit element covered by the exponential map. But, as G is connected, it is generated by any neighborhood of the unit.
- When $(\mathcal{U}, \mathcal{H})$ is a finite dimensional representation of a linear Lie group G then $\mathcal{H}^\infty = \mathcal{H}$. Hence $\mathcal{U}$ is smooth as a mapping of G into $GL(\mathcal{H})$. In this paper we will be focusing on finite dimensional representations.

Properties of Auto-Adjoint Infinitesimal Generator of the Unitary Representation

Now using the proof's steps above let us transfer the exhibited properties there to the auto-adjoint infinitesimal generator of the unitary representation we will work with. Furthermore, we will add some other known properties we will prove or cite their sources:

$$A(\beta) = -\imath\beta_{\mathcal{H}} = -\imath\,\mathcal{U}_*(\beta) =: -\imath\frac{d}{dt}\mathcal{U}(\exp(t\beta))|_{t=0}, \tag{25}$$

1. From the relation (19) we infer that for $m \in \mathcal{H}$:
$$\mathcal{U}(\exp(t\beta))m = \exp(t\beta_{\mathcal{H}})m \tag{26}$$
2. The right-hand side of the previous equation is solution to the ordinary differential equation
$$m'(t) = \beta_{\mathcal{H}} m(t) \tag{27}$$
with the initial condition $m(0) = m$.
3. For any function $\phi \in \mathcal{A}(G)$ and from the relations (23) and (24) we obtain:
$$\begin{aligned} \beta_{\mathcal{H}}\mathcal{U}(\phi) &= \mathcal{U}(L_\beta\phi) \\ \mathcal{U}(\phi)\beta_{\mathcal{H}} &= -\mathcal{U}\left(\left(R_\beta + \mathrm{Tr}(ad\beta)\right)\phi\right). \end{aligned} \tag{28}$$
4. The auto-adjoint infinitesimal generator $-\imath\beta_{\mathcal{H}}$ is symmetric:
$$\langle -\imath\beta_{\mathcal{H}}x, y\rangle = \langle x, -\imath\beta_{\mathcal{H}}y\rangle. \tag{29}$$
for all $x, y \in \mathcal{H}$.

5. Using the results from the book of Marsden and Ratiu (Marsden & Ratiu, 2003) (paragraph 9.3) the auto-adjoint infinitesimal generator $-\imath\beta_{\mathcal{H}}$ satisfies
$$[-\imath\xi_{\mathcal{H}}, -\imath\eta_{\mathcal{H}}] = [\xi, \eta] \tag{30}$$
6. Applying the adjoint presentation we have the following important property:
$$\left(Ad_g\beta\right)_{\mathcal{H}} = \mathcal{U}_g\beta_{\mathcal{H}}\mathcal{U}_{g^{-1}} \tag{31}$$
Indeed
$$\begin{aligned}\left(Ad_g\beta\right)_{\mathcal{H}} &= \frac{d}{dt}\mathcal{U}\left(\exp\left(tAd_g\beta\right)\right)\Big|_{t=0} = \frac{d}{dt}\mathcal{U}\left(\exp\left(tg\beta g^{-1}\right)\right)\Big|_{t=0} \\ &= \frac{d}{dt}\mathcal{U}(g)\mathcal{U}\left(\exp(t\beta)\right)\mathcal{U}(g^{-1})|_{t=0} \\ &= \mathcal{U}(g)\frac{d}{dt}\mathcal{U}\left(\exp(t\beta)\right)|_{t=0}\,\mathcal{U}(g^{-1}) \\ &= \mathcal{U}(g)\beta_{\mathcal{H}}\mathcal{U}(g^{-1})\end{aligned}$$
7. We have another important property:
$$[\xi, \eta]_{\mathcal{H}} = [\xi_{\mathcal{H}}, \eta_{\mathcal{H}}] \tag{32}$$
Indeed, recall the known relation:
$$[\xi, \eta] = \frac{d}{dt}\left(e^{t\xi}\eta e^{-t\xi}\right)|_{t=0}$$
then
$$\begin{aligned}[\xi, \eta]_{\mathcal{H}} &= \left(\frac{d}{dt}\left(e^{t\xi}\eta e^{-t\xi}\right)|_{t=0}\right)_{\mathcal{H}} \\ &= \frac{d}{dt}\left(\left(e^{t\xi}\eta e^{-t\xi}\right)_{\mathcal{H}}\right)|_{t=0} \quad by\ linearity \\ &= \frac{d}{dt}\left(\mathcal{U}(e^{t\xi})\eta_{\mathcal{H}}\mathcal{U}(e^{-t\xi})\right)|_{t=0} \quad by\ (31) \\ &= \frac{d}{dt}\left(e^{t\xi_{\mathcal{H}}}\eta_{\mathcal{H}}e^{-t\xi_{\mathcal{H}}}\right)|_{t=0} \quad by\ (26)\end{aligned}$$
8. The derivative of $\beta_{\mathcal{H}}$ with the respect to the inverse temperature β has the following expression:
$$\frac{d\beta_{\mathcal{H}}}{d\beta} = \beta_{\mathcal{H}}. \tag{33}$$
Indeed:
$$\begin{aligned}\frac{d\beta_{\mathcal{H}}}{d\beta} &= \frac{d}{d\beta}\left(\frac{d}{dt}\mathcal{U}\left(\exp(t\beta)\right)|_{t=0}\right) = \frac{d}{dt}\left(\frac{d}{d\beta}\mathcal{U}\left(\exp(t\beta)\right)\right)|_{t=0} \\ &= \frac{d}{dt}\left(t\mathcal{U}\left(\exp(t\beta)\right)\right)\Big|_{t=0} = \frac{d}{dt}\left(\mathcal{U}\left(\exp(t\beta)\right) + t\mathcal{U}\left(\exp(t\beta)\right)\right)\Big|_{t=0} \\ &= \beta_{\mathcal{H}}\end{aligned}$$

Lie Groups Thermodynamics Quantities

Recall the expression of the density matrices ρ_β and the partition function we will deal with

$$\begin{aligned} \rho_\beta &= \frac{1}{\psi(\beta)} \exp(-\imath\beta_{\mathcal{H}}) \\ \psi(\beta) &= \mathrm{Tr}\big(\exp(-\imath\beta_{\mathcal{H}})\big) \end{aligned} \tag{34}$$

It is known in statistical physics that all thermodynamic properties can be obtained from the partition function $\psi(\beta)$. Notice that

$$\mathrm{Tr}(\rho_\beta) = 1.$$

In Souriau's Lie Groups Thermodynamics we need other important quantities. The Massieu potential can be defined as follows:

$$\Phi(\beta) := -\log\big(\psi(\beta)\big) \tag{35}$$

Thermodynamic Heat Q

Another interesting quantity in this framework is the thermodynamic heat Q which is the first derivative of the Massieu potential, i.e.,

$$Q(\beta) := D\Phi(\beta) \tag{36}$$

The expression of the thermodynamic heat can be expressed using the derivative of the partition function $\psi(\beta)$ as follows:

$$Q(\beta) = -\frac{d}{d\beta}\log\big(\psi(\beta)\big) = -\frac{\frac{d\psi(\beta)}{d\beta}}{\psi(\beta)}$$

Using the linearity of the trace function we have:

$$\begin{aligned} \frac{d\psi(\beta)}{d\beta} &= \frac{d}{d\beta}\Big(\mathrm{Tr}\big(\exp(-\imath\beta_{\mathcal{H}})\big)\Big) = \mathrm{Tr}\left(\frac{d}{d\beta}\big(\exp(-\imath\beta_{\mathcal{H}})\big)\right) \\ &= \mathrm{Tr}\big(-\imath\beta_{\mathcal{H}}\exp(-\imath\beta_{\mathcal{H}})\big) \end{aligned} \tag{37}$$

and using the linearity of the trace function again and the linearity of the infinitesimal generator (33) we obtain:

$$\begin{aligned} \frac{d^2\psi(\beta)}{d\beta^2} &= \frac{d}{d\beta}\mathrm{Tr}\big(-\imath\beta_{\mathcal{H}}\exp(-\imath\beta_{\mathcal{H}})\big) \\ &= \mathrm{Tr}\left(-(\beta_{\mathcal{H}}^2 + \imath\beta_{\mathcal{H}})\exp(-\imath\beta_{\mathcal{H}})\right) \end{aligned} \tag{38}$$

From the previous discussion we infer the following lemma:

Lemma 6. *The thermodynamic heat can be expressed using the derivative of the partition function $\psi(\beta)$ as follows:*

$$Q(\beta) = -\frac{\frac{d\psi(\beta)}{d\beta}}{\psi(\beta)} = \frac{\mathrm{Tr}(-\imath\beta_{\mathcal{H}}\exp(-\imath\beta_{\mathcal{H}}))}{\mathrm{Tr}(\exp(-\imath\beta_{\mathcal{H}}))} \quad (39)$$

and the first derivative of the heat can be expressed as follows:

$$\frac{dQ(\beta)}{d\beta} = \frac{\psi'^2(\beta)}{\psi^2(\beta)} - \frac{\psi''(\beta)}{\psi(\beta)} = \frac{\mathrm{Tr}^2(-\imath\beta_{\mathcal{H}}\exp(-\imath\beta_{\mathcal{H}}))}{\mathrm{Tr}^2(\exp(-\imath\beta_{\mathcal{H}}))} - \frac{\mathrm{Tr}\left(-(\beta_{\mathcal{H}}^2+\imath\beta_{\mathcal{H}})\exp(-\imath\beta_{\mathcal{H}})\right)}{\mathrm{Tr}(\exp(-\imath\beta_{\mathcal{H}}))} \quad (40)$$

Definition 7. *Recall that in quantum statistical systems using the concept of the density matrix then in equilibrium the expectation value of a physical observable is given by*

$$\langle O\rangle_{\rho_\beta} = \mathrm{Tr}(\rho_\beta O)$$

Let ξ be an element in the Lie algebra $\mathfrak{g}$ then the expectation value of $\imath\xi$ in the quantum state ρ_β is given as follows:

$$\langle \imath\xi_{\mathcal{H}}\rangle_{\rho_\beta} = \mathrm{Tr}(\rho_\beta \imath\xi_{\mathcal{H}}) = \langle Q(\beta), \imath\xi\rangle. \quad (41)$$

Indeed:

$$\langle \imath\xi_{\mathcal{H}}\rangle_{\rho_\beta} = \mathrm{Tr}(\rho_\beta \imath\xi_{\mathcal{H}}) = \mathrm{Tr}\left(\rho_\beta \imath \frac{d}{dt}\mathcal{U}(\exp(t\xi))|_{t=0}\right)$$

<u>The Generalized Heat Capacity K</u>

The generalized heat capacity is the symmetric tensor field K, defined as minus the Hessian matrix of the Massieu potential, namely

$$K(\beta) := -D^2\Phi(\beta) = D^2\log\psi(\beta). \quad (42)$$

The generalized heat capacity is computed as

$$K(\beta)(\delta\beta_1, \delta\beta_2) = -D^2\Phi(\beta)(\delta\beta_1, \delta\beta_2) = \mathrm{Tr}(\rho_\beta \imath(\delta\beta_1)_{\mathcal{H}}(\delta\beta_2)_{\mathcal{H}}) - \mathrm{Tr}(\rho_\beta \imath(\delta\beta_1)_{\mathcal{H}})\mathrm{Tr}(\rho_\beta \imath(\delta\beta_2)_{\mathcal{H}}) \quad (43)$$

thereby giving the covariance of the observables $\imath(\delta\beta_1)_{\mathcal{H}}$ and $\imath(\delta\beta_2)_{\mathcal{H}}$ in the quantum state ρ_β. In (Nencka & Streater, 1999), K is called the Bogoliubov-Kubo-Mori metric and chosen as the quantum version to the Fisher metric.

Entropy and Equivariance Properties

The von Neumann entropy s of the density matrix can be expressed in terms of Massieu Potential Φ and the heat Q as follows

$$-\mathrm{Tr}(\rho_\beta\log\rho_\beta) = \mathrm{Tr}(\rho_\beta \imath\beta_{\mathcal{H}}) + \log(\psi(\beta)) = \langle Q(\beta), \beta\rangle - \Phi(\beta) = s(\nu), \quad (44)$$

for

$$\nu = Q(\beta) = \langle \imath(\cdot)_{\mathcal{H}}\rangle_\beta \in \mathfrak{g}^*$$

This is analogue to the result of Lemma 1 in (Barbaresco & Gay-Balmaz, 2020) giving the entropy as the Legendre transform of $\Phi(\beta)$, thus giving a quantum version of the Clairaut equation.

In the quantum case the adjoint representation is defined as follows:

$$\left(Ad_g\beta\right)_{\mathcal{H}} = \mathcal{U}_g\beta_{\mathcal{H}}\mathcal{U}_{g^{-1}}. \tag{45}$$

Using $\left(Ad_g\beta\right)_{\mathcal{H}}$, we can infer the following equivariance properties, which are obtained as in Proposition 5 (Barbaresco & Gay-Balmaz, 2020),

$$\psi\left(Ad_g\beta\right)_{\mathcal{H}} = \psi(\beta) \tag{46}$$

$$\Phi\left(Ad_g\beta\right)_{\mathcal{H}} = \Phi(\beta) \tag{47}$$

$$\rho_{\left(Ad_g\beta\right)_{\mathcal{H}}} = \mathcal{U}_g \circ \rho_\beta \circ \mathcal{U}_{g^{-1}} \tag{48}$$

$$Q\left(Ad_g\beta\right)_{\mathcal{H}} = \left(Ad^*_{g^{-1}}Q(\beta)\right)_{\mathcal{H}} \tag{49}$$

$$s\left(\left(Ad^*_{g^{-1}}\nu\right)_{\mathcal{H}}\right) = s(\nu) \tag{50}$$

$$K\left(\left(Ad_g\beta\right)_{\mathcal{H}}\right)\left(\left(Ad_g\delta\beta_1\right)_{\mathcal{H}}, \left(Ad_g\delta\beta_2\right)_{\mathcal{H}}\right) = K(\beta)(\delta\beta_1, \delta\beta_2), \tag{51}$$

for every $g \in G$.

- *The entropy as Casimir function and Kähler foliation of the dual algebra* $\mathfrak{g}^*$

Let us define the Lie-Poisson bracket on $\mathfrak{g}^*$

$$\{f, g\}(\tau) = \left\langle \tau, \left[\frac{\delta f}{\delta \tau}, \frac{\delta g}{\delta \tau}\right]\right\rangle, \tag{52}$$

where $\mathfrak{g}$ and $\mathfrak{g}^*$ were identified using the duality pairing

$$\langle \tau, \beta \rangle = \mathrm{Tr}(\tau^*\beta) \tag{53}$$

and view $\mathfrak{g}$ as a Lie subalgebra of $\mathfrak{u}(\mathcal{H})$.

Theorem 8. *From the previous equivariance properties one can see that the partition function* ψ *and Massieu potential* Φ *are constant on adjoint orbits. Furthermore the entropy* s *is a Casimir for the Lie-Poisson bracket on* $\mathfrak{g}^*$ *which foliates into the coadjoint orbits. Moreover the coadjoint orbits are the level sets of the entropy and possess a Kähler structure.*

The last results generalize the one in (Nencka & Streater, 1999) where the authors studied the Lie algebras $\mathfrak{so}(3)$ and $\mathfrak{sl}(2, \mathbb{R})$.

Proof. We will summarize the proof in the following bullet points:

- From the equations (46) and (47) we see that the partition function ψ and Massieu potential Φ are constant on adjoint orbits.
- From the equation (50) one can infer easily that coadjoint orbits are the level sets of the entropy. Furthermore using the following result ((Laurent-Gengoux & Pichereau, 2013), Proposition 7.7) that says Casimir functions are the Ad^*-invariant functions, we can conclude the Casimir property of the entropy s.
- It is known that the symplectic leaves of a Lie-Poisson structure are exactly the coadjoint orbits of the coadjoint action when the Lie group G is

connected. Therefore the dual-algebra $\mathfrak{g}^*$ foliates into coadjoint orbits. Using the results discussed in the next section "The canonical Kähler structure for coadjoint orbits of compact Lie groups" of this article we can infer then the Kähler structure.

Furthermore this duality pairing implies that

$$Ad^*_{g^{-1}} = Ad_g \text{ and } ad^*_\beta \tau = [\tau, \beta].$$

This property yields to the following facts:

- Adjoint and coadjoint orbits are identified.
- The Kirillov-Kostant-Souriau symplectic form on coadjoint orbits has the followingh expression:

$$\omega_{\mathfrak{O}}(\tau)\big(ad_\xi \tau, ad_\eta \tau\big) = \langle \tau, [\xi, \eta] \rangle. \tag{54}$$

In (Barbaresco & Gay-Balmaz, 2020) the authors showed that Bogoliubov-Kubo-Mori metric on (co)adjoint orbits can be expressed as:

$$\begin{aligned} K(\tau)\big(ad_\xi \tau, ad_\eta \tau\big) &= \langle ad^*_\xi Q(\tau), ad_\eta \tau \rangle \\ &= \langle Q(\tau), \big[\xi, [\eta, \tau]\big] \rangle \\ &= \langle \tau, \big[[Q(\tau), \xi], \eta\big] \rangle. \end{aligned} \tag{55}$$

Casimir Dissipation/Production

We recall here the general procedure as introduced in (Barbaresco & Gay-Balmaz, 2020). The general equations for Casimir dissipation/production Equation take into account the Lie algebra which is a subalgebra of $\mathfrak{u}(\mathcal{H})$ with the identification $\mathfrak{g} = \mathfrak{g}^*$. Furthermore we consider a symmetric positive bilinear form given is this case by $\gamma(\beta, \tau) = \mathrm{Tr}(\beta^* \tau)$ Let $\Lambda \neq 0$ be a parameter, $h: \mathfrak{g}^* \to \mathbb{R}$ be a Hamiltonian and $k: \mathfrak{g}^* \to \mathbb{R}$ be a function such that

$$\left[\frac{\delta f}{\delta \tau}, \frac{\delta k}{\delta \tau}\right] \tag{56}$$

Then the general equations for Casimir dissipation/production are written as

$$\frac{d}{dt} f = \{f, h\}(\tau) - \Lambda \left\langle \left[\frac{\delta f}{\delta \tau}, \frac{\delta k}{\delta \tau}\right], \left[\frac{\delta s}{\delta \tau}, \frac{\delta k}{\delta \tau}\right] \right\rangle \tag{57}$$

for every f, with $\{f, h\}$ the Lie-Poisson bracket Equation ([lie_poisson_bracket]). As $ad^*_\beta \tau = [\tau, \beta]$, using the results of Equation (41) in (Barbaresco & Gay-Balmaz, 2020) yield

$$\frac{d}{dt}\tau + \left[\tau, \frac{\delta h}{\delta\tau}\right] = \Lambda\left[\left[\frac{\delta s}{\delta\tau}, \frac{\delta k}{\delta\tau}\right], \frac{\delta k}{\delta\tau}\right] \quad (58)$$

The Canonical Kähler Structure Coadjoint Orbits of Compact Lie Groups

Coadjoint orbits appear in many areas of theoretical physics, for instance in representation theory, geometrical quantization, theory of magnetism, quantum optics. They serve as domains of definition in problems connected with nonlinear integrable equations (so called equations of soliton type). Since these equations have a wide application, the remarkable properties of coadjoint orbits interest not only mathematicians but also physicists. For much of the material in this section I have been guided by the contents of (Bordemann, Forger, & Römer, 1986) (Contreras, Ercolessi, & Schiavina, 2016) (Rieffel, 2009).

Complex and Kähler Manifolds

Definition 9 (Complex Manifold). *A manifold Z that can be mapped on $\mathbb{C}^n$ and with analytic diffeomorphism as compatibility condition between charts is called to be a Complex Manifold. Then, on the tangent bundle TZ we can define the complex structure $J: TZ \to TZ$ such that:*

$$\forall v \in TZ: J(v) = -\imath\, v. \quad (59)$$

And $J^2 = -\mathrm{Id}$.

Definition 10 (Kähler Manifold). *Let K be a real and even-dimensional and manifold with:*

- *a complex structure J such that $J^2 = -\mathrm{Id}$*
- *a closed, non-degenerate two-form ω satisfying:*

  $$\omega(x, Jy) + \omega(Jx, y) = 0 \quad (60)$$

  with $x, y \in TK$. In other words ω is a symplectic structure.
- *a positive $(0,2)$-tensor $g(\cdot,\cdot)$ such that:*
  $g(\cdot,\cdot) =: \omega(\cdot, J(\cdot))$ meaning $g(x, y) =: \omega(x, Jy)$ (61)
  The equation (60) implies that g is symmetric and non-degenerate iff ω is non-degenerate. In this latter case g is a metric.

In this case K is said Kähler Manifold.

From the previous properties we can infer the following:

- The property $J^2 = -\mathrm{Id}$ implies

$$\omega(Jx, Jy) = \omega(x, y); \quad g(Jx, Jy) = g(x, y) \quad (62)$$

- The equation ([kaehler_metric_property]) yields:

$$g(x, Jy) + g(Jx, y) = 0 \qquad (63)$$

- Using equation (61) and substituting $y \mapsto Jy$ implies:

$$\omega = -g\big(\cdot, J(\cdot)\big) \qquad (64).$$

This equation permits to define a Kähler manifold starting from a metric. Moreover, using the same tricks, we could start from g and ω and require that

$$J = g^{-1} \circ \omega$$

is such that $J^2 = -\mathrm{Id}$. In literature (J, g, ω) is said to be a compatible triple.

Coadjoint Orbit at a Distinguished Point $\xi_\bullet \in \mathfrak{g}^*$

Let G be a connected compact semisimple Lie group, with Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}^*$ denote the dual Lie algebra.

Let us consider now a distinguished point $\xi_\bullet \in \mathfrak{g}^*$. The bullet denotes the representative role of $\xi_\bullet$ in considering the class of all linear functionals in $\mathfrak{g}^*$ obtained by acting on $\xi_\bullet$ through the coadjoint-representation of the group G. As a matter of fact we are dealing with the orbit $\mathfrak{O}_{\xi_\bullet} \subset \mathfrak{g}^*$ defined by

$$\mathfrak{O}_{\xi_\bullet} = \{Ad_g^* \xi_\bullet \mid g \in G\}.$$

Consider the subgroup $H_{\xi_\bullet}$ of G that leaves $\xi_\bullet$ fixed, defined by

$$H_{\xi_\bullet} = \{h \in G \mid Ad_h^* \xi_\bullet = \xi_\bullet\}.$$

This subgroup is called stabiliser (isotropy) subgroup of (the point) $\xi_\bullet$. We have then

$$\mathfrak{O}_{\xi_\bullet} \simeq \mathrm{G}/H_{\xi_\bullet} \qquad (65)$$

We denote $\mathfrak{h}_{\xi_\bullet}$ the Lie algebra of the stabiliser of $\xi_\bullet$. The action of G on $\mathfrak{O}_{\xi_\bullet} \subset \mathfrak{g}^*$ is transitive and therefore it is possible to view the group G as a fibre bundle

$$H_{\xi_\bullet} \to G \to \mathfrak{O}_{\xi_\bullet},$$

where the natural projection map, which is a G-equivariant diffeomorphism, is given by

$$\begin{aligned} \pi: G &\to \mathfrak{O}_{\xi_\bullet} \\ g &\mapsto Ad_g^* \xi_\bullet \end{aligned}$$

with fibre above $\xi_\bullet$ isomorphic to $H_{\xi_\bullet}$. By taking the tangent maps we obtain the exact sequence:

$$0 \to \mathfrak{h}_{\xi_\bullet} \overset{\iota}{\hookrightarrow} \mathfrak{g} \overset{d\pi}{\to} T_{\xi_\bullet}\mathfrak{O}_{\xi_\bullet} \to 0 \qquad (66)$$

where the map from $\mathfrak{g}$ to $T_{\xi_\bullet}\mathfrak{O}_{\xi_\bullet}$ is given by the derivative $d\pi \equiv ad^*$ of the projection and $\iota\colon \mathfrak{h}_{\xi_\bullet} \hookrightarrow \mathfrak{g}$ is the inclusion of the stabilizing sub-algebra into $\mathfrak{g}$. A very important consequence of the exact sequence is the fact that

$$\ker(ad^*\xi_\bullet) = \mathfrak{h}_{\xi_\bullet} \tag{67}$$

A torus subgroup as subset in the center of $H_{\xi_\bullet}$

<u>The Killing Form</u>

Let us denote by $\mathfrak{K}$ the negative of the Killing form of $\mathfrak{g}$:

$$\mathfrak{K}(X,Y) = -Tr[ad_X ad_Y]. \tag{68}$$

Then $\mathfrak{K}$ is positive-definite because G is compact. Using the Ad-invariance

$$\mathfrak{K}\left(Ad_g(X), Ad_g(Y)\right) = \mathfrak{K}(X,Y),\ X,Y \in \mathfrak{g},\ g \in G$$

we can infer that the action Ad of G on $\mathfrak{g}$ is by orthogonal operators with respect to $\mathfrak{K}$, and thanks to the ad-invariance

$$\mathfrak{K}(ad_X Y, Z) = -\mathfrak{K}(Y, ad_X Z),\ X,Y,Z \in \mathfrak{g}$$

we obtain that the action ad of $\mathfrak{g}$ on $\mathfrak{g}$ is by skew-adjoint operators with respect to $\mathfrak{K}$.

Because $\mathfrak{K}$ is definite, it is a well known linear algebra result that for $\xi_\bullet \in \mathfrak{g}^*$ there is an $\eta_\bullet \in \mathfrak{g}$ such that

$$\xi_\bullet(X) = \mathfrak{K}(X, \eta_\bullet) \ \text{ for all } X \in \mathfrak{g} \tag{69}$$

Furthermore the Ad-stability (isotropy) subgroup of $\eta_\bullet$ namely

$$H_{\eta_\bullet} = \{h \in G \mid Ad_h \eta_\bullet = \eta_\bullet\}$$

Satisfies

$$H_{\xi_\bullet} = H_{\eta_\bullet} \tag{70}$$

Indeed, we have to prove that

$$\forall h \in G, Ad_h \eta_\bullet = \eta_\bullet \Leftrightarrow Ad_h^* \xi_\bullet = \xi_\bullet$$

We know that by definition

$$\langle Ad_h^* \xi_\bullet, X\rangle = \langle \xi_\bullet, Ad_{h^{-1}} X\rangle = \xi_\bullet(Ad_{h^{-1}} X) = \mathfrak{K}(Ad_{h^{-1}} X, \eta_\bullet)$$

Using the fact that the action Ad of G on $\mathfrak{g}$ is by orthogonal operators with respect to $\mathfrak{K}$ the previous equality leads to:

$$\langle Ad_h^* \xi_\bullet, X\rangle = \mathfrak{K}(X, Ad_h \eta_\bullet)$$

which yields the desired conclusion.

The Torus Subgroup $T_\bullet$

A torus T is a compact connected abelian subgroup of G. A torus T is called maximal if there is no torus properly containing T. One can always obtain a torus subgroup by considering the subgroup $\exp(\mathbb{R}X)$ for $X \in \mathfrak{g}$. We apply these ideas here. Let $T_\bullet$ be the closure in G of the one-parameter group

$$t \mapsto \exp(t\eta_\bullet),\ t \in \mathbb{R}$$

so that $T_\bullet$ is a torus subgroup of G.

In the following we are collecting some properties about $T_\bullet$:

- The stability subgroup $H_{\xi_\bullet}$ consists exactly of all the elements of G that commute with all the elements of $T_\bullet$.
- Note that the torus subgroup $T_\bullet$ is contained in the center of $H_{\xi_\bullet}$ (but needs not coincide with the center).
- Since each element of $H_{\xi_\bullet}$ will lie in a torus subgroup of G that contains $T_\bullet$, it follows that $H_{\xi_\bullet}$ is the union of the tori that it contains, and so $H_{\xi_\bullet}$ is connected (Corollary 4.22 of). Thus for most purposes we can just work with the Lie algebra, $\mathfrak{h}_{\xi_\bullet}$, of $H_{\xi_\bullet}$ when convenient.
  In particular,
  $$\mathfrak{h}_{\xi_\bullet} = \{X \in \mathfrak{g} \,|\, [X, \eta_\bullet] = 0\},$$
  and $\mathfrak{h}_{\xi_\bullet}$ contains the Lie algebra, $\mathfrak{t}_\bullet$, of $T_\bullet$ which results by passing to the Lie algebras in the equality (70).

Orthogonal of the Toral Lie Algebra

Using the results in let $\mathfrak{m} = \mathfrak{h}_{\xi_\bullet}^\perp$ be the orthogonal subspace of $\mathfrak{h}_{\xi_\bullet}$ in $\mathfrak{g}$ with respect to the Killing form $\mathfrak{K}$. Since Ad preserves $\mathfrak{K}$, then $\mathfrak{m}$ is carried into itself by the restriction of Ad to $H_{\xi_\bullet}$. Thus

$$[\mathfrak{h}_{\xi_\bullet}, \mathfrak{m}] \subseteq \mathfrak{m}.$$

It has been seen in the exact sequence (66), that $\mathfrak{m}$ can be conveniently identified with the tangent space to $T_{\xi_\bullet}\mathfrak{O}_{\xi_\bullet}$ at the coset $H_{\xi_\bullet}$ (which corresponds to the point $\xi_\bullet$ of the coadjoint orbit). In the next section a Kähler structure will be defined for $\mathfrak{m}$.

Complex Structure on $\mathfrak{m}$

The first component of the Kähler structure we will need is a symplectic form $\omega_\bullet$. This is the Kirillov- Kostant-Souriau form, defined initially on $\mathfrak{g}$ by

$$\omega_{\bullet}(X,Y) = \xi_{\bullet}([X,Y]) = \mathfrak{K}([X,Y],\eta_{\bullet}) = \mathfrak{K}(Y,[\eta_{\bullet},X]). \tag{71}$$

Because $\eta_{\bullet}$ is in the center of $\mathfrak{h}_{\xi_{\bullet}}$, we see that

$$\text{if } X \in \mathfrak{h}_{\xi_{\bullet}} \text{ then } \omega_{\bullet}(X,Y) = 0 \ \forall\, Y \in \mathfrak{g}.$$

Conversely,

$$\text{if } X \in \mathfrak{g} \text{ and if } \omega_{\bullet}(X,Y) = 0 \ \forall\, Y \in \mathfrak{g}$$
$$\text{then, as } \mathfrak{K} \text{ is nondegenerate, } [X,\eta_{\bullet}] = 0, \text{ so that } X \in \mathfrak{h}_{\xi_{\bullet}}.$$

Thus $\omega_{\bullet}$ "lives" on $\mathfrak{m}$ and is nondegenerate there. Because Ad preserves $\mathfrak{K}$ and $H_{\xi_{\bullet}}$ stabilizes $\eta_{\bullet}$, it is easily seen that the restriction of Ad to $H_{\xi_{\bullet}}$ preserves $\omega_{\bullet}$, that is,

$$\omega_{\bullet}\left(Ad_g(X), Ad_g(Y)\right) = \omega_{\bullet}(X,Y),\ \forall X,Y \in \mathfrak{m},\ \forall g \in H_{\xi_{\bullet}}.$$

As proposed in (Rieffel, 2009) we follow the proof of Proposition 12.3 of (Cannas da Silva, 2001) in order to construct a complex structure on $\mathfrak{m}$. Thanks to the nondegeneracy of $\mathfrak{K}$, there is a unique linear operator, $\Gamma_{\bullet}$, on $\mathfrak{m}$ such that

$$\omega_{\bullet}(X,Y) = \mathfrak{K}(\Gamma_{\bullet}X,Y),\ \forall X,Y \in \mathfrak{m}. \tag{72}$$

From equation (71) one can infer that $\Gamma_{\bullet}$ is $ad_{\eta_{\bullet}}$ restricted to $\mathfrak{m}$ and so $\Gamma_{\bullet}$ is skew-symmetric, that is,

$$\Gamma_{\bullet}^{\dagger} = -\Gamma_{\bullet}.$$

This fact has been also cited in the proof of Proposition 12.3 of (Cannas da Silva, 2001). Furthermore using the nondegeneracy of $\mathfrak{K}$ we get that $\Gamma_{\bullet}$ is invertible and as $\eta_{\bullet}$ is in the center of $\mathfrak{h}_{\xi_{\bullet}}$, the Ad-action of $H_{\xi_{\bullet}}$ commutes with $\Gamma_{\bullet}$. Let us define the polar decomposition of $\Gamma_{\bullet}$ as follows:

$$\Gamma_{\bullet} = |\Gamma_{\bullet}| J_{\bullet}. \tag{73}$$

Since $\Gamma_{\bullet}$ is invertible, so are $|\Gamma_{\bullet}|$ and $J_{\bullet}$ which yields that $J_{\bullet}$ is an orthogonal transformation with respect to the Killing form $\mathfrak{K}$. Because $\Gamma_{\bullet}$ is skew-symmetric, so is $J_{\bullet}$, so that

$$J_{\bullet}^{-1} = J_{\bullet}^{\dagger} = -J_{\bullet},$$

and $J_{\bullet}$ commutes with $|\Gamma_{\bullet}|$. In particular,

$$J_{\bullet}^{2} = -\mathrm{Id}, \tag{74}$$

where Id denotes the identity operator on $\mathfrak{m}$. This means exactly that $J_{\bullet}$ is a complex structure on $\mathfrak{m}$, preserved by the Ad-action of $H_{\xi_{\bullet}}$.

Kähler Structure Definition and Ad-action of $T_\bullet$ on $\mathfrak{m}$

The final piece of structure is a corresponding inner product, $g_\bullet$, on $\mathfrak{m}$, defined by

$$g_\bullet(X,Y) = \omega_\bullet(X,J_\bullet Y) = \mathfrak{K}(\Gamma_\bullet X, J_\bullet Y) = \mathfrak{K}(|\Gamma_\bullet|X,Y) \quad (75).$$

Clearly $g_\bullet$ is positive-definite, and is preserved by the Ad-action of $H_{\xi_\bullet}$ but we still need a reasonably explicit expression for the Riemannian metric corresponding to $g_\bullet$.

<u>Ad-action of $T_\bullet$ on the Lie Algebra $\mathfrak{m}$</u>

Let us start by examining the Ad-action of $T_\bullet$ on the Lie algebra $\mathfrak{m}$. By means of $J_\bullet$ we make the Lie algebra $\mathfrak{m}$ into a $\mathbb{C}$-vector space, by defining $\imath X$ to be just $J_\bullet X$ for $X \in \mathfrak{m}$. When we view $\mathfrak{m}$ as a $\mathbb{C}$-vector space in this way we will denote it by $\mathfrak{m}_{J_\bullet}$. Since the Ad-action of $H_{\xi_\bullet}$ (and thus of $T_\bullet$) on $\mathfrak{m}$ commutes with $J_\bullet$, this action respects the $\mathbb{C}$-vector space structure. Define a $\mathbb{C}$-sesquilinear inner product, $\mathfrak{K}_\bullet^{\mathbb{C}}$ on $\mathfrak{m}$ by

$$\mathfrak{K}_\bullet^{\mathbb{C}}(X,Y) = \mathfrak{K}(X,Y) + \imath\,\mathfrak{K}(J_\bullet X, Y). \quad (76)$$

It is linear in the second variable. The Ad-action of $H_{\xi_\bullet}$ on $\mathfrak{m}_{J_\bullet}$ is unitary for this inner product. The Ad-action of $T_\bullet$ on $\mathfrak{m}_{J_\bullet}$ then decomposes into a direct sum of one-dimensional complex representations of $T_\bullet$, whose corresponding representations of $\mathfrak{t}_\bullet$ are given by real-linear functions on $\mathfrak{t}_\bullet$ whose values are pure-imaginary (the "weights" of the ad-action). We let $\Delta_\bullet$ be the set of real-valued linear functionals α on $\mathfrak{t}_\bullet$ such that $\imath\alpha$ is a weight of the ad-action. It will be convenient for us to set, for each real-linear real-valued functional α on $\mathfrak{t}_\bullet$,

$$\mathfrak{m}_\alpha = \{X \in \mathfrak{m}_{J_\bullet} \mid ad_\eta(X) = \imath\alpha(\eta)X = \alpha(\eta)J_\bullet X \ \forall\, \eta \in \mathfrak{t}_\bullet\}. \quad (77)$$

This yields to:

$$\mathfrak{m}_\alpha = \{0\} \text{ exactly when } \alpha \notin \Delta_\bullet.$$

For any $X \in \mathfrak{m}_\alpha$ and $Y \in \mathfrak{m}_{J_\bullet}$, one infers from equation (71) that

$$\begin{aligned} g_\bullet(X,Y) &= \omega_\bullet(X,J_\bullet Y) \\ &= \mathfrak{K}([\eta_\bullet,X],J_\bullet Y) = \mathfrak{K}(\alpha(\eta_\bullet)J_\bullet X, J_\bullet Y) \\ &= \alpha(\eta_\bullet)\mathfrak{K}(X,Y). \end{aligned} \quad (78)$$

Thus for $\alpha \in \Delta_\bullet$ and $X \in \mathfrak{m}_\alpha$ with $X \neq 0$ we have

$$0 < g_\bullet(X,X) = \alpha(\eta_\bullet)\mathfrak{K}(X,X), \quad (79)$$

and therefore

$$\alpha(\eta_\bullet) > 0.$$

The above discussion permitted M. A. Rieffel to obtain the following proposition describing $|\Gamma_\bullet|$ (Rieffel, 2009):

Proposition 11. *For each $\alpha \in \Delta_\bullet$ the restriction of $|\Gamma_\bullet|$ to $\mathfrak{m}_\alpha$ is*

$$\alpha(\eta_\bullet)\mathrm{Id}_{\mathfrak{m}_\alpha},$$

where $\mathrm{Id}_{\mathfrak{m}_\alpha}$ is the identity operator on $\mathfrak{m}_\alpha$. In particular, $\alpha(\eta_\bullet) > 0$ and on $\mathfrak{m}_\alpha$ we have

$$g_\bullet = \alpha(\eta_\bullet)\mathfrak{K}.$$

If P_α denotes the orthogonal projection of $\mathfrak{m}$ onto $\mathfrak{m}_\alpha$, then

$$|\Gamma_\bullet| = \sum_{\alpha\in\Delta_\bullet} \alpha\,(\eta_\bullet)P_\alpha.$$

This proposition shows how strongly dependent $g_\bullet$ is on the choice of $\xi_\bullet$. In contrast, different $\xi_\bullet$'s that give $\eta_\bullet$'s that generate the same group $T_\bullet$ may have the same subspaces $\mathfrak{m}_\alpha$.

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