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Some Thoughts on Teaching Axiomatic Set Theory

The Zermelo-Fraenkel axioms for set theory are the ones presented in many textbooks on this subject[1]. There is another, less well-known axiomatization of the concept of set that comes from category theory. From a purely logical or syntactical point of view, the main difference lies (quite misleadingly) in which concepts each axiomatization takes as basic or undefined. There has been much discussion among philosophers as to whether category theory should replace Zermelo-Fraenkel set theory as a foundation for mathematics. However, they seem to have paid little attention to the category-theoretic approach to set theory[2], which comes in the form of a particular category, viz. the category of sets and maps between them, and which is also an axiomatic set theory. In this paper I consider three fundamental mathematical concepts (functions, sets and natural numbers) and place them side by side within Zermelo-Fraenkel set theory and the category of sets and maps between them. The main aim is to offer teachers and students alike, alternative ways of thinking critically about the world of sets.

Keywords: set, function, natural number, successor function, category theory

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[1]In the bibliography, I have included only a brief and somewhat random selection of introductory textbooks on set theory. The interested reader can easily find a great number of them through an internet search.
[2]After the publication of Lawvere FW, Rosebrugh R (2003), Leinster T (2014) and McLarty C (2017) are, to my knowledge, among the few exceptions.

Introduction

The concepts of *set*, *function* and *number* permeate mathematics. One of the aims of this paper is to show how the concepts of *function* and *natural number* are defined within Zermelo-Fraenkel axiomatic set theory (hereafter ZF) solely in terms of this theory's basic notions of *set* and *membership*, where the latter is a relation between sets. Another purpose is to show how the concepts of *element* and *natural number* are axiomatized within the framework of category theory. More specifically, I will present some of the axioms of a special category, *viz.* the category of sets, whose basic notions are those of *function* or *map*, and *set*. The first category-theoretic axiomatization goes back to 1964[3] but I will base my presentation on a reformulation from 2003 due to F. William Lawvere and Robert Rosebrugh[4] (hereafter LR). The ZF axioms I will present here are standard and can be found in almost any textbook on axiomatic set theory.

Contrasting these two accounts opens up a discussion concerning how to best teach the mathematical concepts of *set* and *function*, since each approach has its own advantages and disadvantages when it comes to learning axiomatic set theory. As it is the case in many mathematical branches, both approaches make extensive use of definitions and axioms. So I will present only those definitions and axioms from each account that are necessary for the discussion, keeping technical details and proofs to a minimum.

Although there is certainly much more to ZF and LR than what I discuss here, I see these axiomatic theories as two remarkable mathematical inquiries into the world of sets, and I hope to raise enough interest among teachers and students to motivate them to learn more about set theory.

The Intuitive Concept of *Function*

Functions in mathematics are means for correlating elements of a given set with elements of some of other set. But there are also functions that correlate elements of a set, let us say A, with elements of A itself, for example, the *identity function* on A. This function assigns to each element a of A, the element a. From an intuitive point of view, it seems clear that identity functions on any set always exist, as the example apparently has shown. However, when one works within an axiomatic theory, one has to be careful and not take for granted everything that seems an "obvious" truth. Indeed, the LR approach takes as some of its axioms several seemingly obvious truths about functions, such as the existence of identity functions for all sets. The following are other properties that mathematicians use in practice and, like in the case of identity functions, they seem obviously true. But before stating them, we need some special notation.

[3]Lawvere FW (1994).
[4]Lawver FW, Rosebrugh R (2003).

In practice, we say that a function, let us call it *f*, "goes from" some set *A* "to" some set *B* (not necessarily different from *A*). This directionality is expressed with the notation

$$f: A \longrightarrow B$$

or

$$A \xrightarrow{f} B$$

For expressing that *a* is an element of *A* we use the ZF notation and write $a \in A$. The idea behind the intuitive concept of *function* is that *for each* $a \in A$, there is *one and only one* element $b \in B$ correlated with $a \in A$ *via f*. Thus we write $f(a)=b$. This correlation could be given by some rule or it could be completely arbitrary. As an example of a function given by a rule we have the case of the identity function on an arbitrary set *A*. If we denote this function by $id_A: A \to A$, then $id_A(a)=a$ for each $a \in A$. If *A* and *B* are sets, there may not be any functions from *A* to B or there may be just one or even infinitely many, it all depends on which sets *A* and *B* are. But practice shows that identity functions always exist.

Formally, in order for a correlation *f* between the elements of *A* and the elements of *B* to be a function, the following two requirement must be satisfied

(1) for each $a \in A$, there exists an element of *B* denoted by $f(a)$; and
(2) for any $a, a' \in A$, if $a=a'$ then $f(a)=f(a')$.

Item (1) tells us that each element of *A* is correlated *via f* with at least one element of *B*. And item (2) says that *f* assigns to each element of *A* *exactly one* element of *B*.

It can easily be verified that identity functions indeed satisfy (1) and (2).

Let us now look at another seemingly true fact about functions. Let *A*, *B* and *C* be sets, and $f: A \to B$ and $g: B \to C$ be functions. Consider $a \in A$. Then $f(a)$ is an element of *B* so we can apply the function *g* to it and obtain an element $g(f(a))$ of *C*. Thus by combining in this way the functions *f* and *g*, we obtain a correspondence between the elements of *A* and the elements of *C*. In other words, item (1) above is satisfied. This correspondence from *A* to *C* is denoted by $gf: A \to C$ and it is called *f followed by g* or the *composition* of *f* followed by *g*. Let us now verify that item (2) is also satisfied. Let then $a, a' \in A$ be such that $a=a'$. Since by hypothesis $f: A \to B$ is a function, then $f(a)=f(a') \in B$. Since $g: B \to C$ is also by hypothesis a function, then $g(f(a))= g(f(a'))$. The result of this process is denoted by

$$gf: A \to C$$

In other words, for any $a \in A$, $gf(a) = g(f(a))$. What we have proved is then, that if $a = a'$, then $gf(a)=gf(a')$, and so $gf: A \to C$ is also a function. In general, if $f: A \to B$ is a function, the set A is called its *domain* and the set B its *codomain*. So we have just seen that for any functions $f: A \to B$ and $g: B \to C$ (that is, functions such that the codomain of one coincides with the domain of the other) there is a function with domain A and codomain C called their composition and denoted by $gf: A \to C$. There may of course be other functions from A to C but we have just seen that *given* any functions $f: A \to B$ and $g: B \to C$, they determine a specific function from the domain of f to the codomain of g, called *the* composition of f followed by g.

The LR approach begins here. It takes as its primitive notions those of *map* (or *mapping*), *domain* and *codomain*. The axioms regulating the concept of *map* in LR are presented here as expressing what I have been calling some seemingly obvious truths about functions. So the existence of *composition* for any appropriate pair of maps is an axiom, as is the existence of an identity maps for all sets. However, composition and identity maps are *not* defined as we have done above in terms of their values at each element of their respective domains. In particular, identity functions are characterized axiomatically by their behavior with respect to composition. And composition is characterized axiomatically by a property mathematical practice shows it has, as we will shortly see.

In order to see what these two axioms state, we need a criterion for determining whether two given functions *with the same domain and the same codomain*, say $i: A \to B$ and $j: A \to B$, are the same function or not. Needless to say, this criterion also stems from mathematical practice and it is the following

$$\text{For any } a \in A, \text{ if } i(a)=j(a), \text{ then } i=j.$$

We will now use this criterion to show a special property all identity functions have. So let $f: A \to B$ be a function, $id_A: A \to A$ the identity function on A and $a \in A$. Then consider the composition $fid_A: A \to B$ and let us calculate its value at $a \in A$

$$fid_A(a)=f(id_A(a))=f(a)$$

If the above criterion were true, we would then be justified in concluding that the composition $fid_A: A \to B$ is the same function as $f: A \to B$. And the same reasoning would apply for the identity on B $id_B: B \to B$ and the composition $id_Bf; A \to B$, namely, that $id_Bf = f$. It is the equalities $fid_A = f$ and $id_Bf = f$ that constitute the axiom in LR characterizing identity maps, for any sets A and B and any map $f: A \to B$. Another way of expressing this axiom is by saying that identity maps are neutral with respect to composition on the right *and* on the left.

Let us now suppose that we are given sets A, B, C and D and functions

$$f: A \to B,\ g: B \to C \text{ and } h: C \to D$$

We can then compose them and obtain the functions $h(gf): A \to D$ and $(hg)f: A \to D$. Then, for any $a \in A$

$$h(gf)(a)=h(gf(a))=h(g(f(a)))=hg(f(a))=(hg)f(a)$$

Again, if the criterion for equality between functions (with the same domain and the same codomain!) were true, we would be justified in concluding that $h(gf)=(hg)f$. But of course, we cannot use this criterion in LR since we do not (yet[5]) have at hand the concept of *element* in this theory. Although a version of this criterion is also an axiom in LR, this latter theory "privileges" maps and composition over elements, son in LR it is the equality $h(gf)=(hg)f$ what characterizes axiomatically the composition of any maps

$$f: A \to B,\ g: B \to C \text{ and } h: C \to D$$

The equality $h(gf)=(hg)f$ is then an axiom in LR and it is called *associativity of composition*.

Thus so far we have seen that LR takes as axioms some basic features about functions that stem from mathematical practice but without using the concept of *element*.

First Interlude: *Functions* and One-Element Sets

In this section we will take a close look at the relationship between one-element sets and functions *from* and *to* them, all in the context of mathematical practice. So let us suppose A is a one-element set, say $A=\{a\}$, and let B be an arbitrary but non-empty set. If $f: B \to \{a\}$ is a function, then f *must* assign the only element a of A to each $b\in B$, there simply is no other option. So, once again, mathematical practice shows that for any one-element set A and any non-empty set B, there is exactly one function from B to A. Notice that it does not matter what the element a is, the crucial point is that it is A's only element. The case in which B is empty involves various subtleties that would take us too far afield. Suffice it to say here that even when B is empty, there is also exactly one function from B to any one-element set[6].

Let us now consider functions *from* a one-element set $\{a\}$ to an arbitrary but non-empty set B. Since B is non-empty, we can take any one of its elements. So let us suppose we take $b\in B$. We can now define a function f_b: $\{a\} \to B$ as $f_b(a)=b$. And we can do this for *each* $b\in B$. So it seems that there are as many functions *from* any one-element set to B as there are elements in B. Notice however, that if B were empty, there could not be any functions from a one-element to B[7]. It is precisely this close relationship between the elements of a (non-empty) set B and functions from any one-element set to B, what can help in understanding the definition of *element* in LR. However, we clearly need first to have at hand a characterization of one-

[5]For the definition, see the section entitled ***Elements* in LR** below.

[6]Indeed, there is one and only one function from *the* empty set (for, as I explain in the section entitled **Natural Numbers in ZF** below, there is only one empty set in ZF) to *any* other set. It is called *the inclusion* function of the empty set into the set in question.

[7]Given the way in which I have explained the intuitive concept of *function*, I think it is not difficult to see why in fact there are no functions from *any* set to the empty set.

element sets that does not involve the notion of *element*. For this, LR uses the special property one-element sets have with respect to any set *B* that we saw in the previous paragraph namely, that there is exactly one map from *B* *to* any set with just one "element". It may seem that there is a circularity involved that cannot be avoided. Fortunately, this is not the case, as we will see in the next section.

Elements in LR

We have seen thus far that LR is primarily about maps and their composition. But in a somewhat derivative way it is also about the domains and codomains of these maps. Many of its axioms postulate the existence of certain sets but *exclusively* in term of maps *from* and/or *to* them. One of these axioms postulates the existence of at least one set, denoted by 1, with the property that for any given set *A*, there is exactly one map from *A* *to* 1. The uniqueness of this map is usually expressed by a dotted line

$$A \dashrightarrow 1$$

and the symbol for it is $!_A : A \dashrightarrow 1$.

We can now raise the question concerning the existence of maps from 1 to 1. Clearly, there is at least one such map: the identity on 1. But moreover, since 1 is the codomain of this map, there can be no other maps from 1 *to* 1. So we can conclude that the identity on 1 is the only map from 1 to 1. If we now define an *element* of a set *A* as a map *from* 1 to *A*, it is clear that the set 1 has just one element. It may seem strange to think of elements as maps, but I hope that the discussion in the previous section makes this definition of *element* not completely unnatural. At this point, it may be useful to think of LR as a language with its special vocabulary of maps and composition and into which we are attempting to translate things that we are used to express in mathematical practice in terms of the membership relation. So let us now see how we can recast in the language of LR the idea of evaluating a function $f: A \to B$ at a given element $a \in A$.

Since in LR an element *a* of *A* is a map $a: 1 \to A$, we can compose it with $f: A \to B$ and obtain a map *from* 1 to *B*

$$fa: 1 \to B$$

which clearly is an element of *B*. So in LR, to evaluate a map $f: A \to B$ at a given element

$$a: 1 \to A$$

of *A*, is to compose this element with *f*. In other words, evaluation of a map at a given element of its domain is just a special case of composition. In symbols: $f(a) = fa$, where the expression on the left of the equality symbol comes from ordinary mathematical practice and the intuitive concept of function. Since composition is uniquely determined then, any map

$$f: A \to B$$

and any given element a: $1 \to A$ of A, determine uniquely an element of B, namely,

$$fa: 1 \to B$$

This element is also called *the value of f at a*. In this way, we recover the intuitive idea of functions as ways of correlating elements from a set with elements of another set: the element of B that a map f: $A \to B$ correlates with a given element a: $1 \to A$ of its domain, is precisely the composition fa: $1 \to B$.

Since in LR *elements* are maps with domain 1, it is clear that identity maps

$$id_A: A \to A$$

evaluated at any element a: $1 \to A$ of their corresponding domains, give as a result that same element, because the axiom characterizing identity maps states that they are neutral with respect to composition. In symbols: $id_A(a) = id_A a = a$.

Let us now consider the case of evaluation of a composite gf at an element a of its domain. Since evaluation is a special case of composition and since composition is associative, we have the following

$$(gf)(a) = (gf)a = g(fa) = g(f(a))$$

This shows that evaluation of a composite is a special case of the associativity of composition.

As mentioned in the previous section, the criterion used in mathematical practice to determine whether two given functions are the same or not, is an axiom in LR. Now that we have recast the concept of *element* in term of *maps*, we can state this axiom and it is the following

> Let f: $A \to B$ and g: $A \to B$ be any maps. If, for any a: $1 \to A$, the composition fa: $1 \to B$ is the same map as the composition ga: $1 \to B$, then $f = g$.

In summary, by taking the notions of *map* and *composition* as basic, LR takes as some of its axioms certain properties which, as we have, seem unquestionably true about the intuitive notion of function.

The theory developed by Lawvere and Rosebrugh does not, of course, stop here. In particular, it postulates the existence of sets other than 1, and always in terms of maps to and/or from them. There are axioms postulating the existence of, for example, finite products and finite sums, as well as the existence of a set that turns out to not have elements. But the axiomatic characterization of certain special sets solely in terms of maps to and/or from them, has an important consequence which will be relevant when we discuss the ZF approach. We will use the axiomatic characterization of the set 1 as an example of this important feature of LR, but first we need a definition.

Definition

Let A and B be any sets. We say that they are *isomorphic*, and write $A \cong B$, if there are maps

$f: A \to B$ and $g: B \to A$ such that $gf = id_A$ and $fg = id_B$.

Let us now suppose that there is a set A with the same property as 1, namely, that for any set C, there exactly one map from C to A. Then in particular, there is exactly one map from 1 to A, let us call it h: 1 ----- A. Since there is also a (unique!) map $!_A$: A ---- 1, we can compose them $!_A h$: $1 \to 1$. But, as we saw earlier, the only map from 1 to 1 is the identity id_1 on 1. Therefore $!_A h = id_1$. Analogously, and since by hypothesis for any set C there is only one map from it *to A*, then there is only one map from *A to A*. But both the identity on A and the composition $h!_A$: $A \to A$ are maps *to A*. Hence, $h!_A = id_A$, and therefore $A \cong 1$. So *any* set with the same axiomatic property as 1 is isomorphic to 1. This is sometimes expressed by saying that 1 is characterized *uniquely up to isomorphism* or that 1 *is unique up to isomorphism*. Recall that the axiom characterizing 1 stated that there is *a* set with a certain property expressed solely in terms of the existence of certain maps; it did *not* state that there is a *unique* set with a certain property. This contrasts sharply with the ZF approach, for in this theory sets are characterized by specifying what their members are and there is an axiom stating that any set is uniquely determined by its members. In the next two sections I present a particularly illuminating and important example of this contrast.

Natural Numbers in ZF

Zermelo-Fraenkel set theory has a remarkable feature: from the postulation of a set that has no members and with the aid of a few axioms, a quite large number of non-empty sets can be proved to exist. Moreover, these sets can be placed in a certain "order" so that, when taken as a whole, the resulting set bears a strong resemblance to the familiar sequence of natural numbers

$$0, 1,2\ 3, \ldots, n, n+1, \ldots$$

This is sometimes expressed by saying that "the" natural numbers *can be constructed* in ZF. But since there are various options for constructing the natural numbers, and since there is no mathematical criterion for privileging one option over the others[8], it is perhaps more accurate to say that ZF offers us a method for recasting

[8]Benacerraf P (1965) famously argued that (natural) numbers cannot be sets, precisely because the theory cannot tell us what sets exactly they are.

in various ways the pre-theoretical concept of natural number in terms of the membership relation. In the next section we will see how LR recasts in terms of maps what all these options have in common, thus characterizing uniquely up to isomorphism the pre-theoretical concept of *set of natural numbers*. But let us first take a brief look at the ZF method for characterizing the sequence of natural numbers.

Unlike my account of the first axioms of LR, in which I argued that these axioms express basic properties of the intuitive notion of *function*, my presentation in this section of some of the axioms of ZF is not intended to show that they capture the intuitive or naïve concept of *set* as a collection of "things". With a few exceptions, I think *sets* in ZF are very strange entities, as I will try to illustrate in this section.

Recall that the only basic, undefined notions in ZF are *set* and the relation of *membership*, which is a relation between sets. So in ZF, the elements of a set are always sets themselves. Everything in ZF is a set and some sets are members of other sets. But strangely enough, the first axiom in ZF postulates the existence of a set that has no members. The second axiom, called the *extensionality axiom*, states that for any sets A and B, if they have the same elements, then they are the same set. In its contrapositive form, this axiom says that for any sets A and B, if $A \neq B$, then one of them has at least one element that is *not* a member of the other set. With the extensionality axiom expressed in this way, it is easy to prove that there is only one set that has no members: if there were two such sets, then one of them should have an element that does not belong to the other set; but there can be no such element, since by hypothesis neither of them has elements. So there a special symbol for *the* set that has no members: $\varnothing$. It is called *the* empty set.

The third axiom states that if A and B are sets, there is a set whose members are *exactly* the sets A and B. So this new set is determined uniquely by A and B, and the notation for it is $\{A, B\}$. With the aid of these three axioms, the existence of many other sets can be proved. Here are some examples:

$$\{\varnothing\}$$
$$\{\varnothing, \{\varnothing\}\}$$
$$\{\varnothing, \{\varnothing, \{\varnothing\}\}\}$$
$$\{\varnothing, \{\varnothing, \{\varnothing, \{\varnothing\}\}\}\}$$

The bracket notation might make it difficult to see, for each of these sets, what sets its elements are. But with some patience, one can see that, with the exception of $\varnothing$ and $\{\varnothing\}$, all those sets that can be formed with the use of the first three axioms, have exactly two elements. Moreover, it seems intuitively clear that there is an infinite number of this kind of two-element sets: we can just keep combining them in pairs and obtain more and more new two-element sets. *If* we could then take this infinitude of sets into *one single* set, we would have a set with an infinite number of elements. But clearly our first three axioms fall short of this, and we thus need a new axiom. However, the way in which ZF introduces an axiom of this kind follows a different (and quite clever) path. There is a good reason for this, as we will shortly see. But first we need two more axioms and a few definitions.

Since in ZF the elements of sets are also sets, we can take what is called their *union*. For example, if $A = \{a, b\}$, then both a and b are sets and the *union of A*, written $\cup A$, is *the* set whose members are the elements of a and/or the elements of b. In this example A has just two elements, so its union $\cup A$ is more commonly written as $a \cup b$. So a set x belongs to $\cup A$ if $x \in a$ and/or $x \in b$. It is easy to prove that if, for example, a is the empty set $\varnothing$, then $\varnothing \cup b = b$. All this constitutes a special case of the union axiom, and although this is all we need for our purposes, we state the axiom in its full form, which is the following

Union Axiom

For any given set A, $\cup A$ is a set.

Let us now consider a set A and a property P that its elements may or may not have. The following axiom states that all the elements of A that do have that property constitute a set. For some properties, there may of course be no elements in A having that property, in whose case the resulting set would be $\varnothing$. For the purposes of this paper, and recalling that in ZF everything is a set, it is enough to express this axiom as follows

> Let A be a set and P a property of sets. Then there exists a set whose elements are exactly those elements of A that have the property P.

In symbols, this axiom states that $\{a \in A \mid a \text{ has property } P\}$ is a set, where the vertical line means *such that*. It is called the *axiom of separation*[9].

Definition

Let A and B be sets. We say that B is a *subset* of A, if every element of B is also an element of A, in whose case we write $B \subseteq A$. If $B \subseteq A$ but A is not a subset of B, we say that B is a *proper subset* of A, and write $B \subset A$.

It is easy to prove that if *both $A \subseteq B$ and $B \subseteq A$*, then $A=B$ (and the converse is also true). But notice that the relation of being a subset is not basic: it is defined in terms of the basic membership relation. If we now use this newly defined relation together with the axiom of separation, we obtain the following general result which will be important for understanding the ZF account of the sequence of natural numbers:

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[9]The axiom of separation is in fact an axiom *schema*. On p.21, Enderton HB (1977), for example, states it as follows: "For each *formula* _ not containing B, the following is an axiom: $\forall t_1 \ldots \forall t_k \forall c \exists B \forall x(x \in B \Leftrightarrow x \in c \;\&\; \_\,)$", my italics. The original idea in Zermelo E (1908a) was, in Zermelo´s own terminology, that for any given set A, and any given *definite property P*, the collection of those and only those elements of A that have property P, always constitutes a set. The problem was, of course, that of determining or characterizing these *definite properties*. A controversy arose and formulations like Enderton's are now the official version of this axiom. In Pallares Vega I (2022) I discuss in more detail this axiom in the context of LR.

For any set A and any property P of sets, $\{a \in A \mid a \text{ has property } P\} \subseteq A$.

We have now arrived at the delicate point in the so-called construction of the natural number sequence. It is called a *sequence* because in it the natural numbers are given in *order* of magnitude, and so the construction has to recast this order in terms of the membership relation or of some other subsidiary notion such as that of *subset*. What we see in the sequence is that if n is a natural number (*i.e.* it is somewhere in the sequence), then there is another natural number immediately following n. This number that appears in the sequence right after n is called *the successor of n* and we will denote it by $s(n)$. So we have that for any natural number n, its successor is strictly larger than, and hence different from n. In symbols

$$n < s(n)$$

The infinitude of the sequence can now be expressed by first, including some set as the first one in the sequence and then, by saying that for *any* n, if n is in the sequence, so is $s(n)$. In other words, if n is a natural number, so is its successor $s(n)$. If we take the empty set as the first one in the sequence, then the latter would look like this

$$\varnothing < s(\varnothing) < s(s(\varnothing)) < \ldots < n < s(n) < s(s(n)) < \ldots$$

Thus far we have seen that sets in ZF, whether they are postulated by an axiom or their existence follows therefrom, are uniquely determined. Here for the first time we encounter an “anomalous” situation: there are many different ways for defining the successor of a natural number, and in such a way that each of them yields a sequence like the one above. The two most common are the following, where n stands for an arbitrary natural number

$$s(n) = n \cup \{n\}$$

and

$$s(n) = \{n\}$$

In the first case we have that $n \subset s(n)$ and in the second one we have that $n \in s(n)$. The natural starting point for the sequence is of course the empty set $\varnothing$. Let us take the first definition of *successor* and see what the first four natural numbers would look like

$0 = \varnothing$
$1 = \varnothing \cup \{\varnothing\} = \{\varnothing\}$
$2 = 1 \cup \{1\} = \{\varnothing\} \cup \{\{\varnothing\}\} = \{\varnothing, \{\varnothing\}\}$
$3 = \{\varnothing, \{\varnothing\}\} \cup \{\{\varnothing, \{\varnothing\}\}\} = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$

And now with the second definition of *successor*, the first four natural numbers would be

$0 = \varnothing$
$1 = \{\varnothing\}$
$2 = \{1\}=\{\{\varnothing\}\}$
$3 = \{2\} = \{\{\{\varnothing\}\}\}$

Many textbooks on set theory give $n\cup\{n\}$ as *the* definition of *successor of n* without presenting or discussing other possibilities, such as the one above. Setting aside the axiom of choice, which poses its own difficulties, almost all the axioms in ZF that postulate the existence of certain sets, are such that these sets are uniquely determined. The axiom introducing the set of natural numbers is one exception, for it requires a specific definition of "the" successor of n for any natural number n. The LR account bypasses this problem in a natural way, since in this theory sets are axiomatized uniquely but always up to isomorphism. And this is what is happening at this point in ZF: there is no unique way for defining "the" successor of a natural number. Therefore, in order to avoid this anomaly, one has to make a choice among all the possible options. Let us just then assume that a choice has been made, although we do not specify which one it is.

In order for the natural numbers to be a set in ZF, we need first a characterization of it in terms of the successor of any given number, since the infinitude of this soon to become a legitimate set in ZF seems to lie in this: there is a first number, any number is different from its successor and whenever n is in the set, so is its successor $s(n)$. This is achieved by the following

Definition

We say that a set I is *inductive* if the following two conditions hold

(1) $\varnothing\in I$; and
(2) For any set x, if $x\in I$, then its successor $s(x)$ is also an element of I.

With this definition, we can finally state the axiom that will serve to "construct" *the* set of natural numbers. But notice that the definition re quires that we first make a choice for defining *the* successor of any element in the set, otherwise the set I will be ill-defined, *i.e.* it will not be a set in ZF. Notice also that from this point on, we only require to assume that a choice has been made, it is not necessary that we specify which one it is.

Axiom of Infinity

There is an inductive set.

One would think that this is *the* candidate for being the set of natural numbers, but it is not, and there is a good reason for this. There are two important properties that natural numbers have and that are widely used in mathematics. The first one is that they allow for proofs by what is called *mathematical induction*. It is a finite method (indeed, it consists of only two steps) for proving properties about *all* natural numbers. In my view, this is one of the great achievements of the so-called construction of the natural numbers, for it *explains* why the method of mathematical induction works for them. The other important property of the natural numbers is that they support the definition of functions by recursion. However, and unlike the case of mathematical induction, the proof in ZF of the recursion theorem is not only everything but straightforward; in addition, it does not have the explanatory power that the proof of the principle of induction does[10]. In contrast, and as we will see in the next section, in LR the axiom of infinity has built into it the recursion theorem. Moreover, the principle of mathematical induction is an immediate consequence of it.

Let us now go back to the natural numbers in ZF. The proposal or construction is an application of the axiom of separation. Let I be an inductive set introduced by the axiom of infinity. As the property P of sets we take "*x is a member of every other inductive set*". The axiom of separation guarantees then the existence of the following set

$$\{x \in I \mid x \text{ belongs to every other inductive set}\}$$

Notice that the axiom of infinity guarantees the existence of at least one inductive set, but that is all that is needed for forming the set above, which is, according to most textbooks on set theory, *the* set of natural numbers: once the set I is chosen, the set of natural numbers is uniquely determined. The crucial point for understanding why the principle of mathematical induction works for the natural numbers lies in the fact that, solely due to the way it has been defined or constructed, the above set is a subset of *every* inductive set. And in ZF this set *is* the set of natural numbers. Let us call it N

$$N = \{x \in I \mid x \text{ belongs to every other inductive set}\}$$

The principle of induction is a theorem in ZF. As a method of proof, it follows directly from the above definition of N. So let us suppose that we want to prove that a certain property P holds for *all* natural numbers, that is, for all elements of N. According to this principle, it suffices to prove two things

(1) $\varnothing$ has property P; and
(2) for any $n \in N$, if n has property P, then its successor $s(n)$ also has property P.

Applying the axiom of separation, we now consider the following set S

[10]See, *e.g.* Enderton HB (1977), pp.73-75. In a footnote on p.74, the author himself acknowledges that "[t]his proof is more involved than ones we have met up to now. In fact, you might want to postpone detailed study of it un after seeing some applications of this theorem." The proof is given in Chapter 4 and it is not taken up again until Chapters 7 and 9.

$$S = \{x \in N \mid x \text{ has property } P\}$$

which, even if it is empty, is clearly a subset of *N*. The key point here is that proving items (1) and (2) above, amounts to proving that the set *S* is inductive, in whose case *N* would be a subset of it. We would then have that *both* $S \subseteq N$ *and* $N \subseteq S$ which, as we have seen, would imply that $N = S = \{x \in N \mid x \text{ has property } P\}$. And the equality $N = \{x \in N \mid x \text{ has property } P\}$ is just another way of saying that *all* natural numbers have property *P*.

Instead of thinking of the equality

$$N = \{x \in I \mid x \text{ belongs to every other inductive set}\}$$

as a definition or construction of "the" natural numbers, I propose to think of it as quite clever proposal of which sets these natural numbers could be or how they might look like in ZF: as we have seen, ZF not only offers us various ways for defining "the" successor of a natural number; in addition, it lacks the means for telling us which definition is the correct one. We could now think of the LR account of natural numbers as somehow saying "it does not matter as long as there is a special element—the first natural number—and a successor *function*" and in such a way that the account guarantees the principle of mathematical induction and the recursion theorem. In the section ***Natural Numbers*** **in LR** we will see how this comes about.

Second Interlude: *Functions* in ZF

In ZF *everything* is a set, including functions. It is indeed a remarkable feature of this theory that it can capture the intuitive notion of *function* from a set *A* to a set *B*, which I think is somewhat dynamic, as a more "static" concept, *viz.* as a specific set. However, the ZF formulation is not only rather intricate, but it is not workable in practice. So in this section I will only present a brief description of how a function $f: A \to B$ is defined in ZF as a particular set constructed from the sets *A* and *B*.

The general idea is first to construct a set, with the aid of some axioms, whose elements are called *ordered pairs* and are denoted as (a, b), where $a \in A$ and $b \in B$. The first axiom that is needed for this construction is the *union axiom* which we saw in the previous section. When applied to two sets *A* and *B*, it guarantees the existence of the set $A \cup B$. The second axiom that is needed for the construction is the following

Power Set Axiom

For any set *A*, there is a set $\mathcal{P}(A)$, called the *power set* of *A*, whose members are exactly all the subsets of *A*.

So we start with two sets *A* and *B*, then take their union $A \cup B$, then apply the power set axiom and obtain $\mathcal{P}(A \cup B)$, *and* then apply again the power set axiom to finally obtain the set

$$\mathcal{P}(\mathcal{P}(A \cup B))$$

A function *f* from *A* to *B* is then going to be defined as a particular subset of $\mathcal{P}$ ($\mathcal{P}$ ($A \cup B$)) and its members are going to be precisely certain ordered pairs (a, b), with $a \in A$ and $b \in B$. However, we are faced here with a problem similar to the one we encountered in the previous section when we discussed the concept of *successor*. What we want an ordered pair to satisfy is, first of all, that for any a, a' $\in A$ and any b, $b' \in B$

$$(a, b) = (a', b') \text{ if and only if } a = a' \text{ and } b = b'$$

and there is more than one way of doing this[11]. But, as it happened in the case of "the" *successor* of a natural number, there is nothing in the theory that forces one option over other possible ones. Here are two different possible choices

$$(a, b) = \{\{\{a\}, \varnothing\}, \{\{b\}\}\} \text{ and}$$

$$(a, b) = \{\{a\}, \{a, b\}\}$$

since both of these sets are elements of the set $\mathcal{P}(\mathcal{P}(A \cup B))$.

Once we choose our definition of *ordered pair*, the next step in the construction is to "extract" from $\mathcal{P}(\mathcal{P}(A \cup B))$ all and only the ordered pairs, and this is done with the aid of the axiom of separation. More specifically, this axiom guarantees the existence of the following set

$$\{x \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid \text{there are } a \in A \text{ and } b \in B \text{ such that } x = (a, b)\}$$

This set is called *the cartesian product* of A and B and it is denoted by $A \times B$. *Any* subset C of $A \times B$ that satisfies the following condition

$$\text{for any } a \in A \text{ there is one and only one } b \in B, \text{ such that } (a, b) \in C$$

will count as function from A to B. So functions in ZF are certain collections of ordered pairs, which in turn are sets with two elements.

Setting aside the anomaly of having more than one option for defining an ordered pair, from this definition of *function*, the basic properties of functions which were taken as axioms in LR, can be proved in ZF. I think this constitutes a great achievement of ZF, but at the expense of conceptualizing functions in a rather cumbersome manner. On the other hand, this approach to functions achieves certain logical economy *vis á vis* the LR approach since in this latter theory, there are also axioms for power sets, cartesian products and a category-theoretic version of the axiom of separation, and yet the basic properties of maps are taken as axioms as well. So it would seem that LR assumes more than is necessary when it comes to

[11]For an interesting discussion of this topic, see Kanamori A (2003), section 5.

the basic properties of functions. But I think this is not the case. I see LR as an extraordinary theory one of whose aims is to inquire into the world of sets as a category, and one of the basic and most important ingredients of any category are always its maps.

Natural Numbers in LR

In order to express in LR that some set *A is equipped with a special* element, we say that *A is given with a specific map from* 1 to *A*, and we give a name to this map, for example

$$a\colon 1 \to A$$

And this is how natural numbers are introduced in LR: as a special set N equipped with a special or "first" element $o\colon 1 \to N$. The anomaly with respect to "the" successor of a natural number that we encountered in ZF, is bypassed in LR because the axiom of infinity in this theory stipulates that N also comes equipped with a specific *map*, called *the successor* map from N to N itself

$$s\colon N \to N$$

Since $s\colon N \to N$ is a map then, for any element $n\colon 1 \to N$ of N, there is a *unique* element of N that we are now justified in calling *the* successor of n: *the* composition $sn\colon 1 \to N$. So N is inductive in the sense of ZF.

But the axiom of infinity in LR says more than just asserting the existence of a set with a specified element and a successor map; or, in terms of ZF, the existence of an inductive set. It tells us how this set, with its given element and successor map, is related to any other set A that also comes equipped with a special element $a\colon 1 \to A$ and some kind of successor map $f\colon A \to A$. One immediate consequence of this is that N, together with its special element

$$o\colon 1 \to N$$

and its successor map $s\colon N \to N$, is unique up to isomorphism. So from the point of view of LR, all the possible ways we encountered in ZF for defining "the" set of natural numbers in terms of successors, are practically the same.

Axiom of infinity

There is a set N together with an element $o\colon 1 \to N$ and a map $s\colon N \to N$, called N's *successor* map, such that for any set A given with both an element $a\colon 1 \to A$ and a map $f\colon A \to A$, there is a *unique map* $h\colon N \to A$ such that

(1) the composition $ho\colon 1 \to A$ is the same map as $a\colon 1 \to A$; and

(2) the composition $hs: N \to A$ is the same map as the composition $fh: N \to A$.

For those not familiar with LR, this axiom might probably seem too abstract and complicated. But what the axiom is telling us solely in terms of maps, is that N together with the two maps $o: 1 \to N$ and $s: N \to N$, *guarantees* the existence of a unique map $h: A \to N$ defined by recursion—items (1) and (2)—for any set A that also comes equipped with a specific element $a{:}1 \to A$ and a map $f{:}A \to A$. And defining maps by recursion is one the important *uses* of natural numbers.

Let us see, for example how items (1) and (2) allow us to calculate the value of

$$h: N \to A$$

at the successor of $o: 1 \to N$, that is, at the map $so: 1 \to N$

$$h(so) = (hs)o = (fh)o = f(ho) = fa$$

So once we have calculated the value of $h: A \to N$ at $n: 1 \to N$ (in the example $n = o$), the axiom tells us how to calculate the value of $h: A \to N$ at the successor sn of n precisely in terms of the given map $f{:}A \to A$ and of $hn: 1 \to N$

$$h(sn) = (hs)n = (fh)n = f(hn)$$

As I mentioned earlier, in ZF the existence of this map of function $h: N \to A$, defined for any set A that comes equipped with a specific element $a{:}1 \to A$ and a function $f{:}A \to A$, is a consequence of the construction of the set of natural numbers. From a logical point of view, it seems reasonable or even desirable that an axiom stipulates the minimum necessary to guarantee the results we want, such as the recursion theorem. But as I previously mentioned[12], the proof in ZF of the recursion theorem is cumbersome and rather convoluted, in sharp contrast to the proof, also in ZF, of the induction principle. So from the point of view of teaching set theory, I find the axiom of infinity in LR preferable to that in ZF[13].

Concerning the principle of induction, LR offers us a slightly different version from the one in ZF since it does not include *properties* in its formulation. Using

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[12]See the section entitled ***Natural Numbers* in ZF**.

[13]See Lawvere FW, Rosebrugh R (2003), p.156 where the authors give a subtle and beautiful argument (quite different from the one I have presented in this section) for introducing the axiom of infinity in the form they do. However, understanding their argument requires a good amount of background in mathematics and in particular in category theory. What I want to emphasize here is that I think their line of reasoning offers us a philosophical explanation of their view on their presentation of LR: axioms and definitions should never be *ad hoc*, they should serve the purpose of teaching, learning and developing mathematics as a whole. Indeed, some of the criticisms to ZF *as a foundation* for mathematics have been to the effect that its account of, for example, the concept of *natural number* or of *function*, are quite unnatural from the point of view of mathematical practice. Be that as it may, I am not arguing neither for nor against ZF or LR as any kind of foundation for mathematics. As I said at the beginning, I think they are both remarkable theories that allow us to inquire axiomatically and critically into various fundamental mathematical concepts.

ZF's terminology, it says that any inductive "subset" of N, must "coincide" with N. As a first approximation, let us start with a "subset" A of N and let us suppose that we have proved the following:

(1) $o\colon 1 \to N$ is an element of A; and
(2) for any $n\colon 1 \to N$, if n is an element of A, then $sn\colon 1 \to N$ is also an element of A.

The principle asserts that we can then conclude that A is isomorphic to N.

As stated, the principle requires the introduction of various new concepts. More specifically, the concept of *part of a set*—which is akin to ZF's concept of *subset*[14]—and the concept of *membership in a part* (of a set). However, recasting in full all of these concepts in the language of category theory falls beyond the scope of this paper, so I will just give a cursory explanation of that of *part*. I will then give a sketchy account of how the axiom of infinity implies the principle of induction in LR.

There are two special kinds of *functions* in mathematics (and *perforce* in both ZF and LR): *injective* and *surjective*. An injective function associates to different elements in its domain, different elements of its codomain. A surjective function f, let us say from A to B, has the property that for every $b \in B$, there is at least one $a \in A$ such that $f(a) = b$. Thus an injective function $f\colon A \to B$ "inserts a copy" of A in B. If $f\colon A \to B$ is surjective, then it "covers" the whole of B; every element of B "comes from", *via* f, at least one element in A. And when a function is both injective and surjective it is called *bijective*. The important point to note here is that if $f\colon A \to B$ is bijective, then there is a function, say $g\colon B \to A$ with the following two properties

$$gf = id_A \quad \text{and} \quad fg = id_B$$

The principle above then says that if items (1) and (2) are satisfied, then there is a bijective map from A to N (or equivalently, from N to A).

Recall that the axiom of extensionality in ZF provides us with a method for proving equality between sets, which is an important relation in ZF. In sharp contrast, not even the relation of being a subset of another set in the sense of ZF, is workable in LR. For consider two sets A and B in LR. If A and B are different sets, then the codomain of every element $b\colon 1 \to B$ of B would always be different from A and hence can never be an element of A, since all the elements of A are maps with codomain A. This observation points towards the need to re-conceptualize the notions of *subset* and membership in a subset. In LR being a subset of another set is defined *via* maps: we say that a set B is a subset or *part* of a set A, if there is an injective map, called *inclusion*, from B to A. This inclusion map is denoted by $i_B\colon B \to A$[15].

[14]For more on the relation between the concepts of *subset* in ZF and *part of a set* in LR, see Pallares Vega I (2022).

[15]In the context of this paper, the definition in LR of *membership in a part*, rather than helping to fully understand items (1) and (2) above, raises issues whose discussion falls beyond the scope

The proof of the induction principle is then roughly as follows. We start with a part i_A: $A \rightarrow N$ of N and assume that both conditions (1) and (2) hold of *this map*. Condition (1), together with the definition of *membership in a part of a set*[16], imply that A has an element, let us call it a: $1 \rightarrow A$. And condition (2) implies that there is map, let us call it f, from A to A

$$f: A \rightarrow A$$

We now have all that is necessary to apply the axiom of infinity, according to which there is a unique map h: $N \rightarrow A$ such that

(1) the composition ho: $1 \rightarrow A$ is the same map as a: $1 \rightarrow A$; and
(2) the composition hs: $N \rightarrow A$ is the same map as the composition fh: $N \rightarrow A$.

And from this, it follows that

$$h\, i_A = id_A: A \rightarrow A \quad \text{and} \quad i_A\, h = id_N: N \rightarrow N$$

and hence that $A \cong N$.

The way in which N was axiomatically introduced, trivially guarantees of course that the map o: $1 \rightarrow N$ is an element of N, and that for any element n of N, its successor

$$sn: 1 \rightarrow N$$

is also an element of N. This, as I mentioned earlier, tells us that N, together with the two maps o: $1 \rightarrow N$ and s: $N \rightarrow N$ is an "inductive set", very much in the sense this concept is defined in ZF. So we can interpret the principle of induction as saying that any part

$$i_A: A \rightarrow N$$

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of this paper. But for the sake of completeness, and of strengthening my point, I include it in this footnote. Let then i: $A \rightarrow B$ be a part of B, and let b: $1 \rightarrow B$ be an element of B. We say that b is an element of i if and only if there is an element a: $1 \rightarrow A$ such that $ia = b$. And we write $b \in i$. As it turns out (because i: $A \rightarrow B$ is injective), the map a: $1 \rightarrow A$ is the only element of A such that $ia = b$. I call the map a: $1 \rightarrow A$ "the element of A witnessing the membership relation $b \in i$". One consequence of the definition of *part* and of *membership in a part* is that, since maps with *domain* 1 are injective, an element a: $1 \rightarrow A$ of any set A is also a part of A. Moreover, any element a: $1 \rightarrow A$ is trivially a member of itself (*qua part* of A) since $(id_A)a = a$, and id_1 is trivially an element of 1 (in fact, it is its only element). So we have that, for any set A and any element a: $1 \rightarrow A$, a: $1 \rightarrow A$ is also a part of A and, moreover $a \in a$. In sharp contrast to this, we have seen that ZF takes good care of distinguishing the subset relation from the basic concept of membership. For an argument concerning the importance of this distinction within ZF, see Kanamori A (2003), section 2.

[16]See previous footnote.

that is also "inductive", must be isomorphic to *N*. Recall that the axiomatic introduction in LR of 1 was made in terms of the unique existence of certain maps, and that one immediate consequence of this was that 1 itself was unique up to isomorphism. As we will now see, something completely analogous happens in the case of *N*. So let us suppose that there is a set *M*, together with an element

$$m\colon 1 \to M \text{ and a map } t\colon M \to M$$

that also satisfy the axiom of infinity. This means that for any set *A* given with both an element $a{:}1 \to A$ and a map $f{:}A \to A$, there is a *unique map* $g\colon M \to A$ such that

(1) the composition $gm\colon 1 \to A$ is the same map as $a\colon 1 \to A$; and
(2) the composition $gt\colon M \to A$ is the same map as the composition $fg\colon M \to A$.

By the axiom of infinity, *N* comes equipped with

$$\text{an element } o\colon 1 \to N \text{ and a map } s\colon N \to N.$$

Hence, by hypothesis there is a unique map $g\colon M \to N$ such that

(a) the composition $gm{:}1 \to N$ is the same map as $o\colon 1 \to N$; and
(b) the composition $sg\colon M \to N$ is the same map as $gt\colon M \to N$

But since by hypothesis *M* also comes equipped with

$$\text{an element } m\colon 1 \to M \text{ and a map } t\colon M \to M$$

then, by the axiom of infinity, there is a unique map $h\colon N \to M$ such that

(c) the composition $ho\colon 1 \to M$ is the same map as $m\colon 1 \to M$; and
(d) the composition $th\colon N \to M$ is the same map as the composition $hs\colon N \to M$

The idea is now to prove the following two equalities between maps

$$id_M = hg\colon M \to M \quad \text{and} \quad id_N = gh\colon N \to N$$

since that would show that $M \cong N$, and hence that *N*—as this set was introduced by the axiom of infinity—, is certainly unique up to isomorphism. More specifically, that would show that any other set with *exactly the same properties as N*, is isomorphic to it.

The proof follows directly from all of our hypothesis about *N* and *M*, namely, the axiom of infinity for both *N* and *M* and items (a) through (d). We start by applying the axiom of infinity to *N* itself. Clearly, the identity $id_N\colon N \to N$ on *N*, is such that $id_N o = o\colon 1 \to N$ and $s id_N = id_N s\colon N \to N$. With the aid of our hypotheses, we will now show that the map

$$gh\colon N \to N$$

—just like the identity map on N—satisfies the equalities

$$(gh)o = o \quad \text{and} \quad s(gh) = (gh)s$$

Besides all of our hypothesis, from this point on, we only need to use the associativity of composition of maps, which is an axiom in LR

$$(gh)o = g(ho) = gm = o$$

$$s(gh) = (sg)h = (gt)h = g(th) = g(hs) = (gh)s$$

But, when applied to N itself, the axiom of infinity asserts that there is only one map from N to N, let us call it j, such that

$$jo = o \quad \text{and} \quad sj = js$$

And we have just seen that *both* maps $id_N\colon N \to N$ and $gh\colon N \to N$ satisfy the above two equalities. Hence $gh = id_N$. A completely analogous reasoning yields the conclusion that the composition $hg\colon M \to M$ is the same map as the identity $id_M : M \to M$. We can then conclude that $M \cong N$.
So in a manner analogous to what happened in the case of 1[17], any set M, equipped with maps

$$1 \to M \to M$$

that also satisfies the axiom of infinity, is necessarily isomorphic to N. The properties with which 1 and N are axiomatically introduced in LR are called *universal*: they tell us that the sets thus introduced enjoy a special status with respect to any other sets similar to them, and the axioms do this by postulating the unique existence of certain maps (either from or to the objects introduced by the axioms) that satisfy certain conditions. As we have seen, sets introduced by means of universal properties are unique up to isomorphism. This contrasts sharply with the examples we have seen from ZF, where sets are introduced by specifying what their elements are, thus making them unique without any further qualification.

Some Afterthoughts on Teaching Axiomatic Set Theory

There is an axiom I did not include in this discussion, the axiom of choice, which was once quite controversial. Just like the axiom of infinity, the axiom of choice is included in LR and in the first axiomatization of set theory which was

[17]Namely, that any set A for which there is one and only one map from any set X to A, is necessarily isomorphic to 1.

given by Zermelo[18]. Nowadays it is customary to distinguish ZF from ZFC, the latter meaning Zermelo-Fraenkel with the axiom of choice, even though the axiom of choice was fundamental to Zermelo. In Pallares-Vega IV (2020), I discuss this axiom placing it too side by side within ZFC and LR, with the main purpose of showing how LR can help us understand why and how the axiom of choice can fail, contrary to what Zermelo thought. The authors of LR call the sets in the category they develop *abstract* or *constant*. This in part serves the purpose of distinguishing this category from others whose sets are less abstract, less constant. They also call the maps in the category of abstract sets *arbitrary* which I take to mean, at least partly, that if $f: A \to B$ is a map in LR then, for *any* $a: 1 \to A$, there always exists a unique $b: 1 \to B$ such that $fa = b$. The authors claim that "[the axiom of choice] is in fact a strong testimony to the 'arbitrariness' of the maps [in the category of abstract sets]"[19]. So LR preserves what for Zermelo was a fundamental feature of what we may now call *abstract* sets. If it is the arbitrariness of the maps, or the presence of the axiom of choice in LR what accounts for the abstractness of its sets then this, together with other axioms that LR and ZF have in common, suggests that both theories constitute different axiomatic inquiries into quite similar, pre-theoretic concepts of *set*. If I am right, then this makes it certainly valuable to teach and learn both approaches to the world of these abstract sets[20].

More than one century separates Zermelo's axiomatization from LR. I believe that the extremely novel character of LR, with its attendant challenges in the context of teaching set theory, supports the view of mathematics as a human activity, and as such, as a part of the larger cultural landscape. I hope to have shown that it is worthwhile to inquiry into this particular area of culture. Looking at mathematics from this non-Platonist perspective, give us hope that there may be other axiomatic set theories yet to be developed and equally worthy of study[21].

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[18]See Zermelo E (1908a).
[19]See Lawvere FW, Rosebrugh R (2003), p. 84.
[20]It should be pointed out, however, that each approach arose for quite different reasons. Concerning ZF there are at least two points of view. According to one of them, the purpose of axiomatizing set theory was to avoid the paradoxes. According to the other one, Zermelo axiomatized set theory with the explicit inclusion of the axiom of choice, in order to defend his proof of the well-ordering theorem. For a thorough defense of this latter view, se Moore GH (2013). Concerning the origins of LR, see Lawvere FW (1964) and Lawvere FW (2005).
[21]Cheng E (2023) is an original and friendly introduction to category theory. Awodey S (2006) is also a good introduction, although it is written in a more traditional style and it requires much more mathematical background than does Cheng E (2023). To my knowledge, Lawvere FW, Schanuel SH (2009), is the first introduction to category in a non-traditional style and aimed at a wide readership like Cheng E (2023).

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