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Improving Variational Characterization Interval Bounds for Gyroscopic Problems

A quadratic matrix polynomial $Q(\lambda) = \lambda^2 I + \lambda B + C$, $B = B^H$, $detB \neq 0$, $C = C^H$, *and* C *is positive definite* $(C > 0)$ *, is gyroscopically stabilized if* $|B| > kl + k^{-1}C$ *holds for some* $k > 0$*, where* $|B|$ *denotes the positive square root of* B^2*. Eigenvalues of a quadratic matrix polynomial are divided into four disjoint intervals: in the first interval there are negative eigenvalues of positive type, in the second interval there are positive eigenvalues of positive type, in the third interval there are negative eigenvalues of negative type and in the fourth interval there are positive eigenvalues of negative type. Therefore, this problem is suitable for the application of the variational characterization method for determining the eigenvalues. The borders of these four intervals are unknown. In this paper we will improve the borders of these itervals, to which we applay the variational characterization. We will consider the application of Sylvester's law of inertia to this thype of problem.*

Keywords: *gyroscopic problems, eigenvalues, variational characterization*

Introduction

Gyroscopically stabilized eigenvalue problems (GSPs) belong to a special class of quadratic eigenvalue problems (QEPs). The QEPs have been the subject of research in mechanics and engineering for a long time due to their wide application in the dynamic analysis of mechanical systems, fluid mechanics and acoustics. Paper [8] summarizes results for different types of nonlinear eigenvalue problems and their application.

A quadratic matrix polynomial

$$Q(\lambda) = \lambda^2 I + \lambda B + C, \;\; B = B^H, \;\; detB \neq 0, \;\; C = C^H > 0 \qquad (1)$$

is gyroscopically stabilized if

$$|B| > kI + k^{-1} C \,, \qquad (2)$$

holds for some $k > 0$, where $|B|$ denotes the positive square root of B^2.

The gyroscopically stabilized eigenvalue problem is

$$Q(\lambda)x = 0, \qquad x \neq 0. \qquad (3)$$

For $G^H = -G$ the free motions of a conservative, time-invariant linear system oscillating about an unstable equilibrium under action of a gyroscopic force are governed by the equation

$$\ddot{u}(t) + G\dot{u}(t) - Cu(t) = 0.$$

Making the substitution $u(t) = xe^{\mu t}$ with x independent of t, and then the rotation of the parameter $\lambda = -i\mu$ leads to the eigenvalue problem

$$Q(\lambda)x = 0,$$

where $B = iG$ is clearly indefinite.

Because of its great importance and simple structure at first sight, this eigenvalue problem is considered often (see e.g. [1,3,4,5,8]).

In the paper [1] it is proven that all eigenvalues of GSP are real and that they are either of positive or negative type. In the paper [3] it is proven that all eigenvalues are located in four disjoint intervals. The authors explain in [3] one decomposition of the space $\mathbb{C}^n$ on two subspaces V and W with properties $V \cap W = \{0\}$ and $dim(V + W) = n$ related to GSP. The Schur complement of GSP and its application were considered in the paper [4].

The paper is organized on the following way: the standard tools for nonlinear eigenvalue problems are presented in Section 2. Earlier results for gyroscopically stable problems are presented in Section 3. In Section 4, through a whole series of theorems, we have shown that the variational characterization can be applied to determine all eigenvalues, regardless of sign and type.

Numerical result is given in Section 5, and in Section 6 we gave conclusion and plans for further research.

Tools for Nonlinear Eigenvalue Problems

We consider the nonlinear eigenvalue problem

$$T(\lambda)x = 0, \qquad (4)$$

where $T(\lambda) \in \mathbb{C}^{n\times n}$, $\lambda \in J$, is a family of Hermitian matrices depending continuously on the parameter $\lambda \in J$ and J is a real open interval that may be unbounded.

There are two important tools for solving nonlinear eigenvalue problem:

- Linearization
- Variational characterization [9]

Linearization

A standard approach for solving a polynomial or rational eigenvalue problem (4) is linearization, i.e. to transform the equation (4) to an equivalent linear eigenvalue problem

$$L(\lambda)X = \lambda GX - HX = 0,$$

of higher dimension, where $G, H \in \mathbb{C}^{2n\times 2n}$, $X \in \mathbb{C}^{2n}$, which can be solved by standard methods. More information of linearization can be found in [4] and [8].

Variational Characterization

The second tool is based on a variational characterization of eigenvalues. In order to generalize the variational characterization of eigenvalues we generalize the Rayleigh quotient. We have the following conditions:

Assume that for fixed $x \in \mathbb{C}^n, x \neq 0$, the scalar real equation

$$f(\lambda; x) := x^H\, T(\lambda)x = 0, \quad (A1)$$

has at most one solution $p(x) \in J$. Then $f(\lambda; x) = 0$ defines a functional p on some subset

$D \subseteq \mathbb{C}^n$, which is called the Rayleigh functional of the problem (4).

Let for every $x \in D$, and every $\lambda \in J$, $\lambda \neq p(x)$,

$$(\lambda - p(x))f(\lambda; x) > 0. \quad (A2)$$

Overdamped Problems

Let us define first the overdamped problems.

Definition 1. If p is defined on $D = \mathbb{C}^n \setminus 0$ then the problem $T(\lambda)x = 0$ is called overdamped.

The notation is motivated by the finite dimensional quadratic eigenvalue problem

$$T(\lambda)x = \lambda^2 Mx + \lambda\alpha Cx + Kx = 0,$$

where M, C and K are Hermitian and positive definite matrices.

Theorem 1. [2,7] Under the conditions $(A1)$ and $(A2)$ an overdamped problem has exactly n eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$, which can be characterized by

$$\lambda_j = \min_{dim V = j} \max_{x \in V \setminus \{0\}} p(x).$$

From Theorem 1 we can see that overdamped problems are suitable for application of the variational characterization.

Sylvester's Law of Inertia

In 1858 J. J. Sylvester had proved the law of inertia, which had a great application in linear algebra and numerical linear algebra. Kostić and Voss have generalized Sylvester's law of inertia for nonlinear eigenvalue problem in [1].

Now we will define a concept of inertia of a Hermitian matrix.

Definition 2. The inertia of Hermitian matrix A is the triplet of nonnegative integers $In(A) := (\pi, \nu, \zeta)$, where π, ν and ζ are the number of positive, negative and zero eigenvalues of the matrix A (counting multiplicities).

Theorem 2. [6] Let $T: J \to \mathbb{C}^{n \times n}$ satisfies the conditions of the maxmin characterization and let $(\pi(\sigma), \nu(\sigma), \zeta(\sigma))$ be the inertia of $T(\sigma)$ for some $\sigma \in J$.

a) If $sup_{x \in D}\, p(x) \in J$ then the nonlinear eigenvalue problem $T(\lambda)x = 0$ has exactly $\nu(\sigma)$ eigenvalues $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_{\nu(\sigma)}$ in J not exceeding σ.
b) Let $\sigma, \tau \in J, \sigma < \tau$, and let $(\pi(\sigma), \nu(\sigma), \zeta(\sigma))$ and $(\pi(\tau), \nu(\tau), \zeta(\tau))$ be the inertia of $T(\sigma)$ and $T(\tau)$, respectively. Then the inequality $\nu(\sigma) \ge \nu(\tau)$ holds, and the nonlinear eigenvalue problem $T(\lambda)x = 0$ has exactly $\nu(\sigma) - \nu(\tau)$ eigenvalues $\lambda_{\nu(\sigma)+1} \ge \ldots \ge \lambda_{\nu(\tau)}$ in the interval (σ, τ).

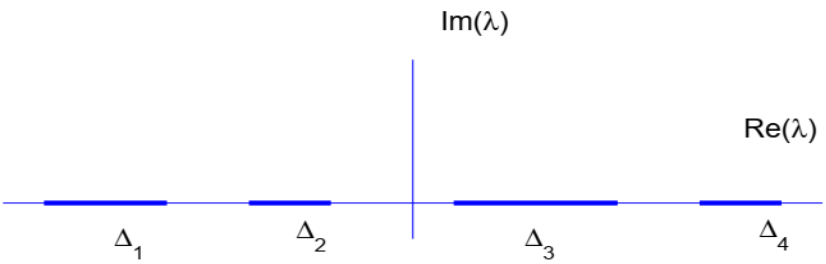
Earlier Results

In this section, we will summarize the previous results.

Theorem 3. [1] The spectrum of a gyroscopic stabilized pencil is real. All eigenvalues are either of positive type (i.e. $x^H Q'(\lambda)x > 0$) or of negative type (i.e. $x^H\ Q'(\lambda)x<0$).

Theorem 4. [3] If $In(B) = (\pi, \nu, 0)$ is the inertia of B, then $Q(\lambda)x = 0$ has 2π negative and 2ν positive eigenvalues. The 2π negative eigenvalues lie in two disjoint intervals, π eigenvalues in each; the ones in the left interval are of negative type, the ones in the right interval are of positive type. The 2ν positive eigenvalues lie in two disjoint intervals, ν eigenvalues in each; the ones in the left interval are of negative type, the ones in the right interval are of positive type.

Figure 1. *Eigenvalue Intervals*



- In Δ_1 there are π negative eigenvalues of negative type;
- In Δ_2 there are π negative eigenvalues of positive type;
- In $\Delta_3$there are $n - \pi$ positive eigenvalues of negative type;
- In Δ_4 there are $n - \pi$ positive eigenvalues of positive type.

There is a basis of eigenvectors corresponding to eigenvalues in $\Delta_1 \cup \Delta_3$, and there is a basis of eigenvectors corresponding to eigenvalues in $\Delta_2 \cup \Delta_4$.

Theorem 5. [3] There exists a π dimensional subspace V of $\mathbb{C}^n$ such that $x^H\ Q(-k_1)x\ <\ 0$ for every $x\ \in\ V\backslash\{0\}$, and for some $k_1 > 0$. There exists a $n - \pi$ dimensional subspace W of C^n such that $x^H\ Q(k_2)x\ <\ 0$ for every $x\ \in\ W\backslash\{0\}$, and for some $k_2 > 0$. Since $\lambda \to x^H Q(\lambda)x$ is a parabola and $x^H Q(0)x\ >\ 0$ it follows at once $V\ \cap\ W\ =\ \{0\}$, i.e. $dim(V\ +\ W)\ =\ n.$

Variational Characterization for Gyroscopically Stabilized Pencils

In order to apply a variational characterization, we need to define certain functions. A desirable property for functionals is that they are easily computable.

The functionals for this class of problems are defined as follows:

Let

$$p_-^+(x) := \begin{cases} p_-(x), & if\ p_-(x) > 0 \\ \infty, & else \end{cases},$$
$$p_+^+(x) := \begin{cases} p_+(x), & if\ p_+(x) > 0 \\ 0, & else \end{cases},$$
$$p_-^-(x) := \begin{cases} p_-(x), & if\ p_-(x) < 0 \\ 0, & else \end{cases},$$
$$p_+^-(x) := \begin{cases} p_+(x), & if\ p_+(x) < 0 \\ -\infty, & else \end{cases}, \qquad (5)$$

Where

$$p_-(x) = -\frac{x^H Bx}{2x^H x} - \sqrt{\left(\frac{x^H Bx}{2x^H x}\right)^2 - \frac{x^H Cx}{x^H x}},$$
$$p_+(x) = -\frac{x^H Bx}{2x^H x} + \sqrt{\left(\frac{x^H Bx}{2x^H x}\right)^2 - \frac{x^H Cx}{x^H x}}.$$

Related to the functional (5) we define the following ranks:

$$J_+^+ := p_+^+(\mathbb{C}^n \setminus \{0\}), \qquad J_-^+ := p_-^+(\mathbb{C}^n \setminus \{0\}), \qquad J_+^- := p_+^-(\mathbb{C}^n \setminus \{0\}),$$
$$J_-^- := p_-^-(\mathbb{C}^n \setminus \{0\}).$$

In earlier researches it was not proved that $max\ J_-^+ < \min J_+^+$ and $max\ J_-^- < \min J_+^-$ hold. Because of that, the problem was to determine the interval to which the variational characterization for the upper functionals can be applied. From the variational characterization of eigenvalues we obtain a Sylvester's theorem for the gyroscopically stabilized quadratic eigenvalue problems in an obvious way.

Main Results

Some of the bounds for variational characterization are given in Theorem 6.

Theorem 6. For $x \neq 0$ let
$f(\lambda;\, x) := x^H Q(\lambda) x = 0$ and $\lambda \geq \sqrt{\lambda_{max}(C)}$. Then $x^H Q'(\lambda) x \geq 0$ holds.

Proof.
Without loss of generality let $\lambda > 0$, then

$$f(\lambda; x) = x^H Q(\lambda) x = 0 \Leftrightarrow x^H Bx = -\lambda\, x^H x - \frac{1}{\lambda} x^H Cx.$$

Hence,

$$\frac{\partial}{\partial \lambda} f(\lambda; x) = 2\lambda x^H x + x^H B x = 2\lambda x^H x - \lambda x^H x - \frac{1}{\lambda} x^H C x$$
$$= \lambda x^H x - \frac{1}{\lambda} x^H C x$$
$$\frac{\partial}{\partial \lambda} f(\lambda; x) \geq 0 \Leftrightarrow \lambda x^H x - \frac{1}{\lambda} x^H C x \geq 0$$
$$\Leftrightarrow \lambda^2 \geq \frac{x^H C x}{x^H x} \text{ i.e. } \lambda^2 \geq \lambda_{max}(C)$$

In $\left(\sqrt{\lambda_{max}(C)}, \infty\right)$ the conditions of the minmax characterization with Rayleigh functional $p_{\ddagger}^{+}(x)$ are satisfied. Hence, there are at most $n-\pi$ eigenvalues which are maxmin values of $p_{\ddagger}^{+}(x)$. If $\sigma \geq \sqrt{\lambda_{max}(C)}$ and $In(Q(\sigma)) = (\pi, \nu, 0)$, then there are exactly ν eigenvalues of $Q(\lambda)x = 0$ in (σ, ∞). Similar results hold for the intervals $\left(-\infty, -\sqrt{\lambda_{max}(C)}\right)$, $\left(-\sqrt{\lambda_{min}(C)}, 0\right)$, and $\left(0, \sqrt{\lambda_{min}(C)}\right)$, and localization of eigenvalues via a Sylvester's law is possible in these intervals as well. Our new research shows that the eigenvalues and eigenvectors of the matrix B play a crucial role when we consider the gyroscopically stabilized eigenvalue problems.

The obtained results in Theorem 6 are not good enough, so we will consider further this problem.

Theorem 7. For each gyroscopically stabilized eigenvalue problem

$$(\lambda^2 I + \lambda B + C)x = 0 \qquad (6)$$

where $x \neq 0,\ B = B^H,\ det\, B \neq 0,\ C = C^H > 0$, and inertia of the matrix B is $In(B) = (\pi, \nu, 0)$, there is a gyroscopically stabilized eigenvalue problem

$$(\lambda^2 I + \lambda B_0 + D)y = 0 \qquad (7)$$

where $y \neq 0$,

$$B_0 = U^H B U = diag(\lambda_{1,B}, \lambda_{2,B}, \dots, \lambda_{n,B}),$$
$$\lambda_{1,B} < \lambda_{2,B} < \dots < \lambda_{n-\pi,B} < 0 < \lambda_{n-\pi+1,B} < \dots < \lambda_{n,B},$$
$$D = U^H C U,$$

and U is unitary matrix. The eigenvalue problem (7) has the same eigenvalues as the eigenvalue problem (6). Eigenvectors of the eigenvalue problems (6) and (7) are connected by equality $y = U^H x$.

Proof.

According to the Schur decomposition of Hermitian matrix B it is clear that there exists a diagonal matrix $B_0 = diag(\lambda_{1,B}, \lambda_{2,B}, \dots, \lambda_{n,B})$ and unitary matrix U such that $B_0 = U^H B U$. Hence,

$$(\lambda^2 I + \lambda B + C)x = 0$$
$$\Leftrightarrow (\lambda^2 I + \lambda U B_0 U^H + C)x = 0$$
$$\Leftrightarrow (\lambda^2 U U^H + \lambda U B_0 U^H + U U^H C U U^H)x = 0$$
$$\Leftrightarrow U(\lambda^2 I + \lambda B_0 + U^H C U)\, U^H x = 0\,.$$

Due to the unitarity of the matrix U follows

$$(\lambda^2 I + \lambda B_0 + U^H C U)U^H x = 0, \quad U^H x \neq 0, \quad y = U^H x.$$

Remark 1. Due to Theorem 7, instead of the problem (6), we will always consider the problem (7), i.e. without loss of generality, we will assume that the matrix B is a diagonal matrix with eigenvalues on the diagonal and

$$e^{(k)} = (0,0,\dots,1,0,\dots,0), k = 1,2,\dots,n,$$
$$e_i^{(k)} = \begin{cases} 1, & for\ i = k, \\ 0, & for\ i \neq k, \end{cases}$$

are corresponding eigenvectors.

Theorem 8. Let $In(B) = (\pi, \nu, 0)$ be the inertia of the matrix B, and let

$$\lambda_{1,B} < \lambda_{2,B} < \cdots < \lambda_{n-\pi,B} < 0 < \lambda_{n-\pi+1,B} < \cdots < \lambda_{n,B}$$

be the eigenvalues of the matrix B, and $e^{(k)}$, $k = 1,2,\dots,n$, corresponding eigenvectors. Let

$$x^{(1)}, x^{(2)}, \dots, x^{(n-\pi)}$$

be eigenvectors of the gyroscopically stabilized eigenvalue problem (7), which are corresponding to the positive eigenvalues of negative type, and

$$y^{(1)}, y^{(2)}, \dots, y^{(n-\pi)}$$

be eigenvectors of the gyroscopically stabilized eigenvalue problem (7), which are corresponding to the positive eigenvalues of positive type, then

$$V = span\{e^{(1)}, e^{(2)}, \dots, e^{(n-\pi)}\} = span\{x^{(1)}, x^{(2)}, \dots, x^{(n-\pi)}\}$$
$$= span\{y^{(1)}, y^{(2)}, \dots, y^{(n-\pi)}\}$$

Proof. Eigenvectors $e^{(k)}$, $k = 1,2,\dots,n$, of the matrix B_0 determine a basis of the space $\mathbb{C}^n$. Each of the eigenvectors of the problem (7) which are corresponding to the positive eigenvalues of negative type can be represented as linear combination of $n - \pi$ vectors $e^{(k)}$, $k = 1,2,\dots,n$. Let

$$V = span\{e^{(k_1)}, e^{(k_2)}, \dots, e^{(k_{n-\pi})}\} = span\{x^{(1)}, x^{(2)}, \dots, x^{(n-\pi)}\},$$

where $k_1, k_2, \dots, k_{n-\pi} \in \{1,2,\dots,n\}$. Similar as in the proof of Theorem 7, we can reformulate the eigenvalue problem (7) into the problem

$$(\lambda^2 I + \lambda B_0' + D')z = 0 \quad (8)$$

where $z \neq 0$ and B_0' is a diagonal matrix which has on the first $n-\pi$ places on the main diagonal eigenvalues of the matrix B, $\lambda_{k_{1,B}} < \lambda_{k_{2,B}} < \cdots < \lambda_{k_{n-\pi,B}}$, and the rest of elements on the main diagonal are the remaining eigenvalues of the matrix B. We can write the problem (8) in the following form

$$\left(\lambda^2 \begin{pmatrix} I_1 & 0 \\ 0 & I_2 \end{pmatrix} + \lambda \begin{pmatrix} B_1' & 0 \\ 0 & B_2' \end{pmatrix} + \begin{pmatrix} C_{11}' & C_{12}' \\ C_{12}^{T\prime} & C_{22}' \end{pmatrix}\right) z = 0,\ z \neq 0 \quad (9)$$

and $B_1' = diag(\lambda_{k_{1,B}}, \lambda_{k_{2,B}}, \dots, \lambda_{k_{n-\pi,B}})$.

The positive eigenvalues of positive type we obtain from the problem

$$(\lambda^2 I_1 + \lambda B_1' + C_{11}')a = 0, \qquad a \neq 0, \quad z = \begin{pmatrix} a \\ 0 \end{pmatrix}. \quad (10)$$

Let $In(B_1') = (\pi_1, \nu_1, 0)$, then the problem (10) must have $\nu_1 = n - \pi - \pi_1$ positive eigenvalues of negative type. However, the problem (10) has the same all positive eigenvalues of negative type as the problem (8), respectively as the problem (7), and there are $n-\pi$ eigenvalues in total. Hence, there must be $\pi_1 = 0$, respectively $B_1' = diag(\lambda_{1,B}, \lambda_{2,B}, \dots, \lambda_{n-\pi,B})$, and

$$V = span\{e^{(1)}, e^{(2)}, \dots, e^{(n-\pi)}\} = span\{x^{(1)}, x^{(2)}, \dots, x^{(n-\pi)}\}.$$

The same holds for the positive eigenvalues of positive type.

Theorem 9. Let $In\ (B) = (\pi, \nu, 0)$ be the inertia of the matrix B, and

$$\lambda_{1,B} < \lambda_{2,B} < \cdots < \lambda_{n-\pi,B} < 0 < \lambda_{n-\pi+1,B} < \cdots < \lambda_{n,B}$$

eigenvalues of the matrix B. Let

$$B_1 = diag(\lambda_{1,B}, \lambda_{2,B}, \dots, \lambda_{n-\pi,B}),$$
$$B_2 = diag(\lambda_{n-\pi+1,B}, \lambda_{n-\pi+2,B}, \dots, \lambda_{n,B}).$$

Then the gyroscopical stabilized eigenvalue problem

$$\left(\lambda^2 \begin{pmatrix} I_1 & 0 \\ 0 & I_2 \end{pmatrix} + \lambda \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} + \begin{pmatrix} C_{11} & C_{12} \\ C_{12}^{T} & C_{22} \end{pmatrix}\right) x = 0, x \neq 0 \quad (11)$$

has the same positive eigenvalues as the problem

$$(\lambda^2 I_1 + \lambda B_1 + C_{11})a = 0, \qquad a \neq 0, \quad (12)$$

and eigenvectors of the problem (11) and (12) are given by the relation $x = \binom{a}{0}$.

Eigenvalue problem (11) has the same negative eigenvalues as the problem

$$(\lambda^2 I_2 + \lambda B_2 + C_{22})b = 0, \qquad b \neq 0, \qquad (13)$$

and eigenvectors are given by the relation $x = \binom{0}{b}$.

Proof. According to Theorem 8 it is clear that each eigenvector which belongs to the positive eigenvalue has the last one π coordinates equal to zero, while at least one coordinate from the first $n - \pi$ coordinates must be different from zero. Hence, each eigenvector which is corresponding to the positive eigenvalue has the following form

$$x = \binom{a}{0}, a \neq 0. \qquad (14)$$

If we include the vector x from (14) to (11) we obtain eigenvalue problem (12). For the eigenvectors that are corresponding to the negative eigenvalues we prove analogously.

Theorem 10. $\max J_-^+(x) < \min J_+^+(x)$, and all eigenvalues in Δ_3 and Δ_4 are minmax and maxmin eigenvalues of J_-^+ and J_+^+ respectively.

Proof. (Theorem 10 can be proved according to Theorem 1, Theorem 3, Theorem 5, Theorem 9 and $In(B_1)$.) According to Theorem 1 and Theorem 2 all eigenvalues are real and they are distributed in four disjoint intervals. The number of positive and negative eigenvalues depend from $In(B_1)$. According to Theorem 9 we can obtain the positive eigenvalues from the gyroscopically stabilized eigenvalue problem (12). Applying Theorem 5 on the problem (12) there exists $n - \pi$ dimensional subspace W of $\mathbb{C}^{n-\pi}$ so that $w^H Q(k_2)w < 0$ for every $w \in W\backslash\{0\}$, and for some $k_2 > 0$ holds. Therefore, $\max J_-^+(x) < \min J_+^+(x)$. According to the previous the conditions of the variational characterization from Theorem 1 hold.

Theorem 11. $\max J_-^-(x) < \min J_+^-(x)$, and all eigenvalues in Δ_1 and Δ_2 are minmax and maxmin eigenvalues of J_-^- and J_+^- respectively.

Proof. The proof of Theorem 11 is analogously to the proof of Theorem 10.

Numerical Example

The following examples illustrate the results of our research. Exact values in following tables are obtained as zeroes of the polynomial det $(Q(\lambda))$. The eigenvalues of positive and negative type are obtained using variational characterization.

Example 1. Let $B = diag([-20, -15, 7, 10]), C = (C_{ij})$, where

$$C_{ij} = \begin{cases} 0.1 * (i+j), & for\ i \neq j, \\ 2, & for\ i = j. \end{cases}$$

Applying Theorem 9, Theorem 10 and Theorem 11 we obtain the eigenvalues, which we can compare with the exact values that are shown in Table 1 and Table 2.

Table 1. *Numerical Results for the Negative Eigenvalues*

Negative (neg. Type)	Exact value	Negative (pos. Type)	Exact value
-9.7976	-9.7978	-0.3533	-0.3381
-6.6978	-6.6982	-0.1514	-0.1498

Table 2. *Numerical Results for the Positive Eigenvalues*

Positive (neg. Type)	Exact value	Positive (pos. Type)	Exact value
0.0931	0.0919	14.8654	14.8655
0.1420	0.1268	19.8995	19.8996

Example 2. Let $B = diag([-400, -375, -200, 150, 278, 300, 350, 390]), C = (C_{ij})$ as follows

$for\ i = 1{:}8$
$for\ j = i + 1{:}8$
$C_{(i,j)} = randn$
$C_{(j,i)} = C_{(i,j)}$
end
end
$for\ i = 1{:}8$
$C_{(i,j)} = 4$
end
$C = C' * C$

Table 3. *Numerical Results for the Negative Eigenvalues*

Negative (neg. Type)	Exact value	Negative (pos. Type)	Exact value
-389.9455	-389.9455	-0.3755	-0.2952
-349.9109	-349.9109	-0.0886	-0.0758
-299.9264	-299.9264	-0.0468	-0.0468

-277.8908	-277.8908	-0.0376	-0.0311
-149.7338	-149.7339	-0.0262	-0.0143

Table 4. *Numerical Results for the Positive Eigenvalues*

Positive (neg. Type)	Exact value	Positive (pos. Type)	Exact value
0.0261	0.0141	199.8885	199.8886
0.1286	0.0430	374.8708	374.8708
0.1543	0.1228	399.9313	399.9313

It has been observed that the errors are larger around zeros, which is explained by the numerical instability of finding zeros of the polynomial $\det(Q(\lambda))$.

Conclusion

The gyroscopically stabilized eigenvalue problems belong to the quadratic eigenvalue problems. They are interesting for mathematical observations due to their structure, and they have many applications. The variational characterization is a very effective way to find eigenvalues. Even if many properties of the gyroscopically stabilized eigenvalue problems were proven, the intervals of application of the variational characterization have not been precisely determined before. We proved that the variational characterization can always be applied for suitable selected functionals. We also managed to reduce the dimension of the problem. We have illustrated the theoretical results with numerical examples. In further research, we will perform numerical experiments for problems of a larger dimensions.

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