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ABSTRACT

There are examples of non-convex polyhedra that show that it is not possible to decompose them into tetrahedra. Opposite, it is known that convex polyhedra can always be divided into tetrahedra. Such process is known as 3-triangulation. Polyhedra topologically equivalent to sphere with p handles, shortly p-toroids, could not be convex. So, here it is investigated possibilities of their 3-triangulations and if some exists, its properties. It is of interest the minimal necessary number of tetrahedra for the 3-triangulation of a 3-triangulable p-toroid. For that purpose we shall develop the concepts of piecewise convex polyhedra and of graph of connection. Also, some interesting examples will be shown.

Keywords: *3-triangulation of polyhedra, toroids, piecewise convex polyhedra*

Introduction

By generalizing the term polygon, in higher dimensions we get a polyhedron and a *d*-dimensional polytope. As it is known, always is possible to triangulate a polygon with *n* vertices. That is the process of dividing it by $n-3$ diagonals into $n-2$ triangles. We can consider generalization of this process into higher dimensions and call it also triangulation, or more specific 3-triangulation, *d*-triangulation. Than, using only the original vertices, for 3-triangulation we divide a polyhedron into tetrahedra and for *d*-triangulation a *d*-polytope into *d*-simplices.

Already in the case of 3-dimensional space, two problems arise. As it shows e.g. the example of Schönhardt's polyhedron (Schönhardt 1928), there is no possibility to triangulate certain non-convex polyhedra. Also, triangulations of same polyhedron may have different numbers of tetrahedra (Edelsbrunner et al. 1990, Sleator et al. 1988, Stojanović 2005, Stojanović 2008). Considering the smallest and the largest number of tetrahedra in triangulation (the minimal and the maximal triangulation), it is shown that such values, linearly, resp. squarely, depend on the number of vertices.

We shall consider here 3-triangulation of the special class of polyhedra, *p*-toroids. Namely, by the term "polyhedron" we usually mean a simple polyhedron, topologically equivalent to a sphere. On the other hand, there are classes of polyhedra topologically equivalent to torus or *p*-torus (sphere with *p* handles). We shall call such polyhedra 1-toroids and *p*-toroids ($p \in N$ is a given natural number), inspired by Szilassi definition (Szilassi 2005) of torus-like polyhedra. He called them toroids. Here term "toroid" will be used as a common name for *p*-toroids for any $p \in N$.

Although toroids are not convex, under certain conditions it is possible to 3-triangulate them. An example of such 1-toroid is the Császár's polyhedron (Bokowski 2005, Császár 1949, Szilassi 2005, Szilassi 2012). It has 7 vertices and it is triangulable with 7 tetrahedra. It was also discussed as a polyhedron without diagonals (Császár 1949, Szabó 1984, Szabó 2009). In (Brehm 1987, Jungerman et al. 1980, Szilassi 2005) some other examples of *p*-toroids are given, while in (Stojanović 2015, Stojanović 2017, Stojanović 2019) some properties of toroids and additional examples are given.

In the present paper, after defining necessary concepts and a brief overview of the previous results, the method for the construction of a *p*-toroid based on a given graph as its graph of connection will be considered. After that, examples of toroids obtained in the introduced way will be given. These examples show that the minimal number of necessary tetrahedra for 3-triangulation is in accordance with the lower limit given in the Theorem formulated in (Stojanović 2019).

Preliminaries

In this section necessary terms and statements will be given.

3-Triangulations of Convex Polyhedra

It is known that the smallest possible number of tetrahedra in the 3-triangulation of a polyhedron with n vertices is $n - 3$. An example of polyhedron that allows this is e.g. pyramid V_{n-1} with $n - 1$ vertices in the basis and the apex, i.e. n vertices in total. 3-Triangulation is as follows: do any 2-triangulation of the basis into $(n - 1) - 2 = n - 3$ triangles. The apex together with each of such triangles make one of tetrahedra in 3-triangulation. The pyramid V_5 and its 3-triangulation are given in Figure 1.

Another example of polyhedron with the same property is the triangular prisms *Π*. It has 6 vertices and it is 3-triangulable with 3 tetrahedra, as Figure 2 shows. If fact, we may consider triangular prisms *Π* as a "pyramid" with a space pentagon in the basis. If vertices of *Π* are A_1, B_1, C_1 in one of the basis and A_2, B_2, C_2 in the other, then we can assume that A_2 is the apex of the prism and $A_1B_1B_2C_2C_1$ is its basis.

Figure 1. *Pyramid* V_5 *and its Triangulation*

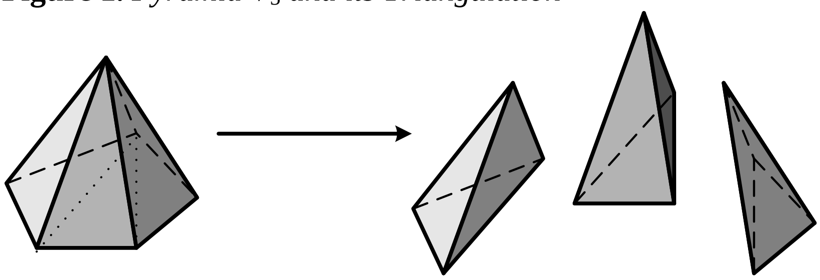
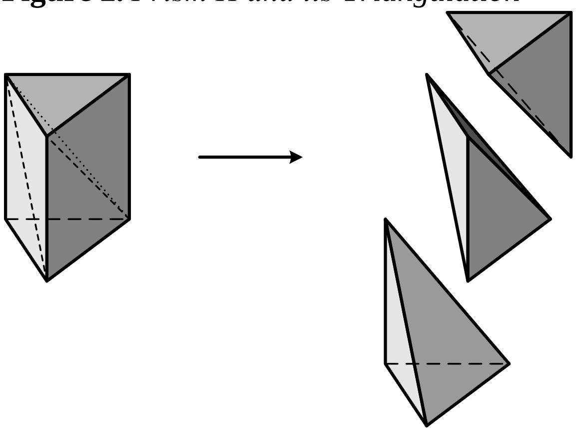


Figure 2. *Prism Π and its Triangulation*



Source: Stojanović 2015.

On the other hand, any 3-triangulation of octahedron - a polyhedron with 6 vertices, gives 4 tetrahedra. 3-Triangulations giving small and especially minimal number ($T_{\min}$) of tetrahedra are examined in (Edelsbrunner et al. 1990, Sleator et al. 1988, Stojanović 2005, Stojanović 2008). In these papers are also given numerous examples of simple polyhedra with $T_{\min} \geq n-3$.

Toroid

First we have to introduce term *p-torus*. In the surface theory it is defined as a cyclic polygon with paired sides. Any side s and its pair S are oppositely directed related to the fixed orientation of the polygon and then glued together. By a standard combinatorial procedure - the polygon can be divided and glued to a cyclic normal form $a_1b_1A_1B_1a_2b_2A_2B_2...a_pb_pA_pB_p$, as a p-torus. This combinatorial procedure is independent of the future spatial placement of the surface. So from any spatial knot (as a topological circle in the space) we can form a torus. Of course, its surface can be 2-triangulated to be a polyhedron.

Based on Szilassi's definition (Szilassi 2005) it is introduced term p-toroid (Stojanović 2019).

Definition 1. *A polyhedron is called p*-toroid, $p \in N$, *if it is topologically equivalent to sphere with p handles* (*p-torus*).

As it was mentioned earlier, here term *toroid* will be used as a common name for all p-toroids.

Piecewise Convex Polyhedron and its Graph of Connection

For needs of our consideration, we shall introduce the following definitions.

Definition 2. *A polyhedron is* piecewise convex *if it can be divided into finitely many convex polyhedra* P_i, $i = 1, ..., m$, *with disjoint interiors. A pair of polyhedra* P_i, P_j *is said to be* neighbouring *if they have a common face called* contact face.

If the polyhedra P_i and P_j are not neighbouring, they may have a common edge e or a common vertex v. That is possible iff there is a sequence of neighbouring polyhedra $P_i, P_{i+1}, ..., P_{i+k} \equiv P_j$ such that the edge e, or the vertex v belongs to each contact face f_l common to P_l and P_{l+1}, $l \in \{i, ..., i+k-1\}$. Otherwise, polyhedra P_i and P_j do not have common points.

Remark 1. *Since a convex polyhedron can be* 3*-triangulated, the same holds for piecewise convex one, especially for a piecewise convex toroid.*

Remark 2. *Each* 3*-triangulable polyhedron is a collection of connected tetrahedra, so it is piecewise convex.*

Definition 3. 1*-toroid is* cyclically piecewise convex *if it is possible to divide it into a cycle of convex polyhedra* P_i, $i = 1, ..., n$, *such that* P_i *and* P_{i+1}, $i = 1, ..., n-1$ *and* P_n *and* P_1 *are neighbours.*

The concept of graph of connection is essential for considerations given in this paper.

Definition 4. *If a polyhedron P is piecewise convex its* graph of connection (*or its* connection graph)*, is a graph with* nodes *representing convex polyhedra* P_i, $i = 1, ..., m$, *the pieces of P, and* edges *representing contact faces between them.*

It is obvious that if a 1-toroid is cyclically piecewise convex, then its graph of connection is a single cycle. Other piecewise convex 1-toroids have graphs with a cycle and additional branches. Similarly, a piecewise convex *p*-toroid form a graph of connection with *p* cycles, and eventually with additional branches.

It is important to mention that division of a polyhedron to convex pieces is not necessarily unique. Since we consider in this paper the minimal number of necessary simplices in 3-triangulation of a toroid *P*, it would be useful to handle with divisions and graphs of toroids in which the minimal 3-triangulation of *P* is in accordance with the minimal 3-triangulation of their pieces, namely not to care about this accordance. It must not happen that the sum of tetrahedra in minimal 3-triangulations of the pieces would be greater than the number of the tetrahedra in 3-triangulation of the whole toroid. So, let us define:

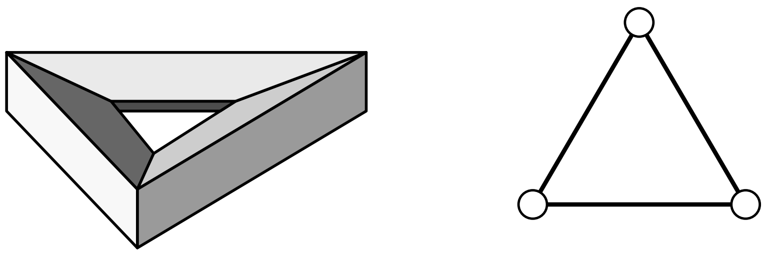
Definition 5. *M*-division *of a polyhedron is a division in which the tetrahedra participating in the minimal* 3-*triangulations of the pieces are at the same time participating in the minimal* 3-*triangulation of the whole polyhedron. A graph of connection of a given polyhedron is m*-graph *if it represents m-division of that polyhedron.*

Remark 3. *We see that m-division and thus m-graph of a polyhedron is not unique. Note that convex pieces of division (m-division) can be either separated tetrahedra or their different collections. Besides that, more possibilities can occur for minimal* 3-*triangulation of the same polyhedron. On the other hand, it is obvious that there exists at least one m-division of a given* 3-*triangulable polyhedron. That is its partition into tetrahedra participating in the minimal* 3-*triangulation.*

An Example of 1-Toroid and Preview of Previously Considered Theorems

One of the examples of cyclically piecewise convex polyhedron T_9 with $n = 9$ vertices is given in (Szillasi 2005). It is composed of three pieces of convex polyhedra which are topologically triangular prisms Π and its property is that four edges meet at each vertex and the faces are quadrilaterals. As it was mentioned earlier, triangular prisms is 3-triangulable with 3 tetrahedra, so we can 3-triangulate T_9 with 9 tetrahedra. Figure 3 shows T_9 and its graph of connection.

Figure 3. 1-*Toroid* T_9 *and its Graph of Connection*



Source: Stojanović 2015.

In earlier papers of author (Stojanović 2015, Stojanović 2017) there were proved the following theorems for 1-toroids and 2-toroids.

Theorem 1. *If a 1-toroid with* $n \geq 7$ *vertices can be 3-triangulated, then the minimal number of tetrahedra in that 3-triangulation is* $T_{\min} \geq n$.

Theorem 2. *If it is possible to 3-triangulate 2-toroid with* $n \geq 10$ *vertices, then the minimal number of tetrahedra for that 3-triangulation is* $T_{\min} \geq n+3$.

In (Stojanović 2019) appropriate theorem for the *p*-toroids was introduced.

Theorem 3. *If a p-toroid with n vertices can be 3-triangulated, then the minimal number of tetrahedra necessary for its 3-triangulation is* $T_{\min} \geq n+3(p-1)$.

From Graph of Connection to *p*-Toroid

In the proofs of Theorem 1 and Theorem 2 (Stojanović 2015, Stojanović 2017) there are discussed different possibilities of connecting pieces in *m*-graph for resp. 1- and 2-toroid. In the proof of Theorem 3 (Stojanović 2019) it was not necessary to consider such possibilities for *p*-toroid. However, for considering the smallest number *n* of vertices in a 3-triangulable *p*-toroid in (Stojanović 2019) it was necessary to take care about possibilities of connecting the pieces in its *m*-graph. There is discussed an example of toroid whose *m*-graph consists of *p*-cycles cyclically arranged, so that form the new, $(p+1)^{\text{th}}$ cycle. Although geometric realization of the toroid introduced in this example is questionable, its combinatorial structure shows us that along with *p* grows the number of possibilities for connecting pieces in *m*-graph.

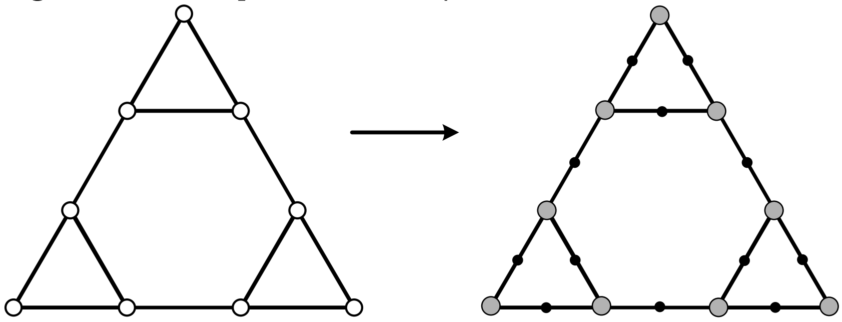
That is the reason to discuss options for *m*-graphs in this section. Also, the consideration of different possibilities of connecting *m*-graphs for *p*-toroids as a consequence opens new questions. The following examples will give the answers to some of them.

Example 1

In accordance with the previous, one of the question is: is it possible to make toroid with 'cycle of cycles' which is geometrically realizable?

If we start with the graph *G* given on the left in Figure 4, and assume that three triangular parts are three 1-toroids T_9 shown in Figure 3, then connections between them are such that all of them must have a common edge in the middle. That means, there are no holes in the center, which would be expected based on the graph *G*. Of course, that means that here we don't have 'cycle of cycles'. Instead of the graph *G* we shall construct new graph G' given on the right in Figure 4. Such construction is possible for each given graph *G* in the following way.

Figure 4. *The Graphs G and G' of 4-Toroid*



To given an arbitrary graph *G*, corresponding graph G' will be formed by splitting each edge of *G* and adding a new node between splitted parts. On the figures of the given examples, the new nodes will be marked black, while the old ones will be marked gray. Each of the graphs G' formed in this way can serve as a connecting graph for a certain polyhedron P'.

Each of the black nodes of G' represent polyhedra of type *Π* in P', while each of the gray nodes *v* of G' represent polyhedra of type V_k, $k \geq 3$, where *k* is the number of edges from *v*. If for some node holds $k = 2$, we can take tetrahedron in appropriate place, i.e. we shall take V_3. If $A_1, A_2, \ldots, A_k$ are vertices in the basis of V_k and *V* is the apex, then contact faces of V_k, $k \geq 3$ and neighbour polyhedra of type *Π* would be $A_i A_{i+1} V$, $i \in \{1, \ldots, k-1\}$, $A_k A_1 V$. In the case $k = 2$, the contact faces will be $A_1 A_2 V$ and $A_2 A_3 V$. In doing so, either polyheda *Π* or V_k can be slightly deformated, if would be necessary.

In this construction inserted prisms *Π* allow the pyramids V_k to be far enough apart to form a handle.

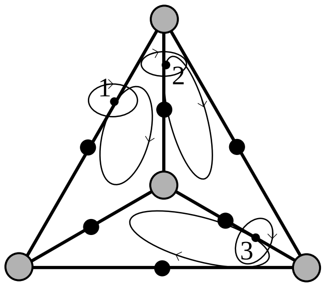
As before, *Π* has 6 vertices and it is 3-triangulable with 3 tetrahedra, while V_k has k+1 vertex and it is 3-triangulable with $k - 2$ tetrahedra. Counting the vertices of P', it is suffices to take into consideration only the vertices of the pyramids of type V_k, because all the vertices of prisms *Π* belong to some of the contact faces.

So, in this example, with graph G' given in the right part of Figure 4, appropriate polyhedron P' consists of 9 tetrahedra V_3 and 12 prisms *Π*. It really have hole in the middle, which means that $p = 4$. Number of the vertices of P' is $9 \cdot 4 = 36$, while the number of tetrahedra in 3-triangulation is $9 \cdot 1 + 12 \cdot 3 = 45$. The estimated minimal number of tetrahedra according to the Theorem 3 is reached, because there $T_{\min} \leq n + 3 \cdot 3 = 36 + 9 = 45$.

Example 2

Another question is: if the graph *G* is the skeleton of some simple polyhedron whether is it right to consider *G* as a “planar” or as a “spherical” one? In the Figure 5 it is given the graph G' of the graph *G* which is skeleton of tetrahedron. It is also marked how to cut appropriate polyhedron P' to determine how many handles it has. As the Figure 5 shows, after 3 cutting of P', the graph G' remains without cycles, i.e. it is tree. This means that P' has 3 handles. Therefore it is right to consider the graphs G and G' as a "planar" and not to take into account "outer face" which surrounds these graphs.

Figure 5. *Tetrahedron as a Graph G' of 3-Toroid*



As in the graph G' the number of edges from each node is equal to 3, particles of P' are 4 tetrahedra V_3 and 6 prisms *Π*. So, P' has $4 \cdot 4 = 16$ vertices and $4 \cdot 1 + 6 \cdot 3 = 22$ tetrahedra in (minimal) 3-triangulation. That is in accordance with the statement of Theorem 3 where for a 3-toroid the minimal number of tetrahedra is $T_{\min} = n + 3 \cdot 2$, in our case $16 + 6 = 22$.

Example 3

It is obvious that a polyhedron P' can be determined in a similar way to any graph G which is the skeleton of a simple polyhedron π. Denote resp. with f, e and v the number of faces, edges and vertices (nodes) of π, with n the number of vertices of P' and with T the number of tetrahedra in 3-triangulation of P'. As in the previous case, the "outer face" is not related to some of the handles of P', i.e. the number of handles of P' is $p = f - 1$. The number n is sum of the vertices of V_{k_i}, $i \in \{1, \ldots, v\}$, for all grey nodes of G'. That means $n = \sum_i (k_i + 1) = 2e + v$, because the sum of the edges from all nodes of π is equal to 2e. Similarly $T = \sum_i (k_i - 2) + 3e = 5e - 2v$, as a sum of the tetrahedra in 3-triangulations of all V_{k_i} and of all *Π*. By the Theorem 3 the minimal estimated value of tetrahedra is $T = 2e + v + 3(f - 2)$. Since according to Euler's theorem holds $f - 2 = e - v$ the minimal value of T in this way is also equal to $5e - 2v$.

If graph G is one of the Platonic bodies, we can also construct appropriate polyhedron P' by the described method. After the tetrahedron for which the results are given in the previous example, the next Platonic body is cube (hexahedron) with $v = 8$, $e = 12$ and $f = 6$. For the corresponding polyhedron P', $p = 5$, number of vertices is $n = 2 \cdot 12 + 8 = 32$ and $T = 5 \cdot 12 - 2 \cdot 8 = 44$. If G is octahedron with $v = 6$, $e = 12$, $f = 8$, then $p = 7$, $n = 2 \cdot 12 + 6 = 30$ and $T = 5 \cdot 12 - 2 \cdot 6 = 48$. In the case of dodecahedron $v = 20$, $e = 30$, $f = 12$, and so $p = 11$, $n = 2 \cdot 30 + 20 = 80$ and $T = 5 \cdot 30 - 2 \cdot 20 = 110$. Finally, if G is icosahedron, then v = 12, e = 30, f = 20, so that p = 19, $n = 2\cdot 30 + 12 = 72$ and $T = 5\cdot 30 - 2\cdot 12 = 126$.

Conclusions

The properties of 3-triangulations for a *p*-toroid when they exist are investigated. It was of interest the minimal required number of tetrahedra for 3-triangulation of *p*-toroid. For these purpose, concepts were developed of piecewise convex polyhedra and of graph of connection. Here was considered method for constructing a *p*-toroid on the base of a given graph as its graph of connection. Especially, more characteristic examples of graphs and corresponding toroids were given. These examples also show that estimated minimal number of tetrahedra in 3-triangulation given in the statement can be reached. Introduced method of constructing *p*-toroid on the base of a given graph gives us opportunity for easier investigating properties of *p*-toroids because of similarities of starting graph and resulting polyhedron, i.e. toroid.

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