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Theories versus the Psychology of Teaching and Learning
Mathematics**

Kakoma Luneta and Paul Giannakopoulos

Athens Institute for Education and Research

8 Valaoritou Street, Kolonaki, 10683 Athens, Greece

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Kakoma Luneta, Associate Professor of Mathematics Education, University of Johannesburg, South Africa

Paul Giannakopoulos, Lecturer, University of Johannesburg, South Africa

**Problem Solving In Mathematics:
Use of Teaching and Learning Theories versus the Psychology of
Teaching and Learning Mathematics**

ABSTRACT

Why problems solving (in general and in Mathematics in particular) is a problem to many people? Does it lie in the teaching and learning situation? The teaching and learning (T & L) of Mathematics has puzzled many researchers, psychologists, educationists and Mathematics educators for decades. If it is accepted that learners have to be taught how to solve problems in Mathematics, then the teaching and learning theory used has to be the appropriate one that fits with Mathematics as well as the context that Mathematics is taught. It is assumed from start that the teachers knows the learners mathematical ability or inability and therefore the knowledge they need to acquire at a given stage. This assumption is one of the many reasons learners fail or go as far as to hate Mathematics. From an early stage mathematics is seen as 'cold and abstract' (the aesthetic and utilitarian aspect of Mathematics). The existing T & L theories (e.g. Behaviourism, Cognitivism, Constructivism), while they appeal to knowledge acquisition, they somehow fall short, especially when it comes to Mathematics. The reason could be because such theories were not conceptualised for the learning of Mathematics but rather for learning in general. However, Richard Skemp in his seminal book the 'Psychology of teaching and learning Mathematics' addressed not only the general learning theories but also how people learn Mathematics. Using Skemp's ideas and pragmatism, a new way of teaching problem solving in Mathematics is suggested, that combines the existing learning theories and the psychology of teaching and learning Mathematics in particular. The paper concludes with a new model for the teaching and learning of Mathematics and problem solving from pre-school to higher education which could assist especially novice Mathematics teachers to improve the teaching and learning situation.

Keywords: t Mathematics, Problem solving, psychology of learning Mathematics

The Nature of Mathematics and the Teaching and Learning of Mathematics

Mathematics can be considered as a very concise and precise subject allowing no room for multiple interpretations or use of redundant words, which means that almost every word or symbol in a mathematical expression conveys meaning. Missing one word or a symbol could lead to misinterpretation or misunderstanding (Zevenbergen, 2001: 17). Mathematics is also perceived by many young and old as ‘cold and abstract’ (it is normally the content that they are referring to) while by few (those who can do Mathematics) ‘hot and abstract’ (Rodd, 2002). The usefulness of Mathematics (utilitarian aim) is denied by few and its usefulness should be promoted from an early stage. However, Mathematics also develops abstract thinking in the learner (aesthetic aim). Here the statement by Skemp (1977: 26) should be highlighted: “Mathematics, is a tool that destroys the human limitations and broadens our mental horizons... an activity of human intelligence which helps the thinker to use Mathematics in all fields.” Thus a pragmatic aspect is added. Adding to that Silver’s (1987) statement that Mathematics is a vibrant, challenging, creative, interesting and constructive subjects, the ‘aesthetic aspect’ cannot be disputed.

Schaffler (1999: 1-6) states that mathematics maybe understood to represent ‘internal’ logical relationships among concepts, or very abstract, though still empirical, generalisations based upon experiences. Mathematical truths are not dependent on experience, though an awareness of them may be suggested by experience.

Finally, Mathematics’ absoluteness, which is based on its inherent constancy, precision, and universality, requires a particular way of teaching which is, paradoxically, culture dependent and place based.

The above dialogue sets the scene for discussing the various existing theories of learning in general and converging to the psychology of T & L of Mathematics of Skemp (1977) where problem solving is the aim and the goal of Mathematics.

Learning Theories

It can be argued that reality resides in everyone (inner) and it is everywhere (outer). The outer reality could be related to behaviourism while the inner to cognitivism. Mathematics is a subject that is manifested from the early stages of a human being by means of problem solving in an attempt to harness reality. Many young children can solve quantity problems long before they go to school through intuition. They have an inner ability for it. To solve any problem, a certain knowledge is necessary. Thus the problem becomes one of acquisition of knowledge or construction of knowledge. In the former the learner is considered as an ‘empty container’ that is filled with the ‘knowledge of the teacher, the one that possesses the knowledge’, while in the latter case the learner constructs his/ her knowledge in his/her cognitive structure. These two examples give rise to Behaviourism and Cognitivism..

Behaviourism (dealing with observable, outer behaviour, knowledge is viewed as a commodity, teacher centred) and Cognitivism (which deals with what is going on in the mind, with the mental activities such as perception, thinking, knowledge representations and memory, learner centred). Hergenhahn and Olson (1997) maintained that neither of the two theories are 'pure'; they are both 'hybrids'; It is a matter which of the two predominates over learning. These two dominant learning theories have given rise to other theories such as functionalism, associationism, constructivism which in turn was approached from personal, radical, critical, social and contextual perspectives. Social constructivism gained ground during Vygotsky's (1978) times where knowledge is viewed a socially constructed normally through collaboration.

According to Skemp (1977), knowledge is the product of a synthesis of various interrelated acquired concepts (irrespective how they were acquired, be it the Behaviouristic or the Cognitive way). The situation normally dictates which of the two theories should be the dominant theory. The concepts and their relationships combined give rise to principles which form the structures of knowledge. Combining knowledge and skills give rise to problem solving.

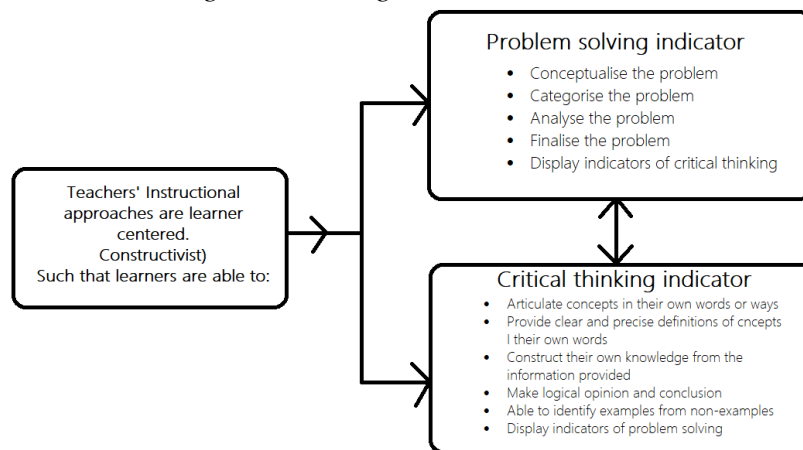
Finally, Pragmatism which stresses the experimental character of the empirical science, emphasises the active phases of the experimentation. Being logical, learning truths that appear self-evident or common sense is not enough. Pragmatism encourages problem thinking, preventing problems from happening. Learning from experiences is an active process. Pragmatism encourages imaginative theorising by the student but at the same time insists upon control of such theorizing by the outcomes of active experimentation (Schaffler, 1999). In fact, Giannakopoulos (2012) introduced the idea of a psycho-pragmatic approach to learning which combines the psychological aspect (a hybrid of Behaviourism and Cognitivism) and Pragmatism which gave rise to the 'Act of Learning' (see Figure 2 later in the paper).

Most research (Sullivan, 2011; Luneta, 2013) in the teaching and learning of mathematics regards problem solving as one of the critical outcomes of learning and appeals to teachers to prioritise the enhancement of learners' problem-solving skills. Schoenfeld (1985) argues that the teaching and learning of mathematics itself should be viewed as a problem solving undertaking, since a problem solving task compels learners to be actively involved in articulating problems and seeking solutions to them. Jonassen (2000) points out that problem solving, as the essence of mathematics teaching, appeals to contemporary learning theories, such as student-centered learning, open-ended learning and problem-based learning. Jonassen recommends instructional approaches, for example the use of authentic cases, simulations, modeling, coaching and scaffolding that appeal to problem-solving outcomes.

We argue in this paper that within a constructivist perspective of instruction, where the learner is at the centre of instruction, problem-solving and critical thinking are viewed as part of the process of instruction. Luneta (2013) for instance points to the direct link between problem-solving and critical thinking by asserting that a problem solver is essentially a critical thinker. Giannakopoulos (2012) for instance recounts that critical thinking can be viewed as the mental

process on the basis of which we make reliable judgments on the credibility of a claim or the desirability of a course of action. Halpern (2003) argues that critical thinking is equivalent to scientific thinking may be described as a higher order thinking, which entails reasonable, reflective and skillful analysis that is focused on deciding what to believe or what to do. Luneta (2013) and Paul, Elder and Bartell (1997) all conclude that critical thinking consists of the mental process of analyzing and evaluating statements or propositions that have been offered as true. It includes a process of reflecting upon the specific meaning of statements, examining offered evidence and reasoning in order to form a judgment and that is the basis of problem-solving. Luneta (2013) provides indicators of problem-solving and critical thinking in a teaching episode and argues that in a constructivist perspective of teaching learners would have acquired the skills of problem solving if they are able to display indicators of critical thinking and illustrates that diagrammatically in figure 1.

Figure 1. *Constructivist Teaching Pivotal in instilling Critical thinking and Problem-Solving Skills among Learners*



In this paper we propose that the indicators can be used in lesson observation to investigate whether or not the teachers’ instructional approaches, evaluation techniques and general dispositions are engaging learner in problem-solving and critical thinking. However effective instructional approaches are such that problem-solving and critical thinking applications in the lesson are interwoven and almost seamless (Luneta 2013). There is a cyclic flow of information from the teacher to the learners that appeal to problem-solving and critical thinking which sometimes is cognitively perceived by both the teacher and the learner. This means that in one activity a learner may perform all the constructs at once intrinsically and the teacher’s instructional approach may appeal to learners’ skills of problem-solving and critical thinking in a single teaching episode. Figure 1 above illustrates the cyclic flow of information as both the teachers and the learners are engaged in instruction and activities that are of problem-solving and critical thinking in nature.

The mathematician best known for his conceptualisation of mathematics as problem-solving, and for his discussions of problem solving strategies in

mathematics, or heuristics, is Pólya; and subsequent works on problem solving is largely based on his work (Schoenfeld 1992:16). In his book, “How to solve it”, Polya (1945:1) contends that learners should be given as much opportunity as possible to work independently, but they cannot make progress in solving problems if they are not given the necessary support by the teacher. Given the consistent emphasis in mathematics publications and curriculum documents on the importance of problem solving in the teaching and learning of mathematics, it could be inferred that the primary goal of mathematics instruction should be to have learners become competent problem solvers (Schoenfeld 1992:2). Reform messages are consistent in their call for reduced emphasis on drilling of computational skills and an increased emphasis on solving problems for which learners have not previously memorised a given procedure. This is the process of problem-solving that research outcomes have advocated for (Schoenfeld 1992; Luneta, 2013; Stein et al 1996). Studies advocate that computation procedures can be developed within the context of solving problems (Hiebert & Wearne 1993:395). Many research studies advocate for the use of problem-solving contexts that employs critical thinking in order to help learners use their prior knowledge to learn essential mathematical relationships and concepts (Osta & Labban 2007:7).

However, unfortunately, the rhetoric of problem solving has not been realised in classroom practice and remains a challenge for teachers (Schoenfeld 1992; Suurtamm & Vézina 2010). As mentioned previously, in most mathematics classrooms learners are not afforded the conditions necessary for the development of their capacity to think and reason mathematically. Furthermore, in almost all mainstream texts, “problem solving” is a separate activity and highlighted as such. The text books, “problem solving” problems are either included intermittently as a recreational activity, or texts contain “problem-solving” sections with drill-and-practice exercises on the strategies they were taught.

Towards a 21st Century Theory of Teaching and Learning

Accepting no behaviour (Behaviourism) is void of thinking, what goes on in the mind of the learner (Cognitivism) and vice versa (we are always thinking of something in a voluntary or involuntary manner) and that knowledge and skills (physical or mental) are necessary to solve problems it is necessary to construct a theory that will take cognizance of all these facts. To achieve this, we look at the T & L situation one that does not result in ‘surface/rote learning’, memory based, but it results in ‘deep learning’, learning with insight, conceptualising (Folk, 2006: 26). For Newton (2001a, Folk, 2006: 26) understanding is both a mental process and a mental product. Here is the teacher, the learner, the environment and the content to be learned. T & L should lead to construction of conceptual knowledge, knowledge that is gained by synthesising various interrelated concepts; a web of concepts.

Conceptual knowledge is one of the types of knowledge that is necessary to solve problems. The others being, declarative (facts/information), procedural

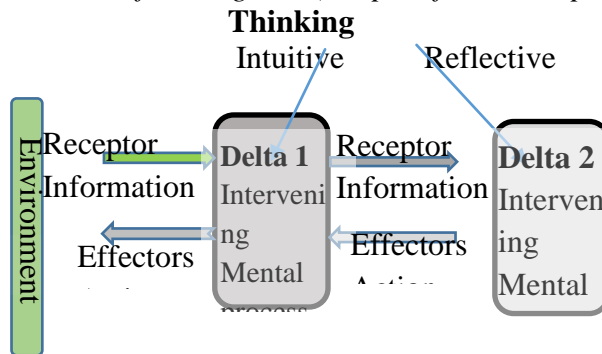
(algorithmic), schematic, strategic, situated, meta-cognitive, situational (context) and conditional knowledge. Since knowledge is the result of principles (concepts and their relationships) then conceptual knowledge could be considered the most important type of knowledge to be acquired. However, in Mathematics and in real life problems all types of knowledge are necessary to a greater or a lesser extent depending on the situation. ‘Knowing that’ (declarative), ‘knowing how’ to use it (procedural), ‘knowing why’ should we use it (schematic), ‘knowing when, where and how’ to use it (strategic) (Shavelson, Ruiz-Primo, & Wiley; 2005), the situation that is applicable (situated) and under what conditions to use it (conditional) could lead to solution of a problem. It can be argued that in real life problems all types of knowledge should be available simultaneously. Mathematics problems are very close to real life problems since as we move higher in education, Mathematics becomes more abstract thus abstract thinking dominates. And using all types of knowledge simultaneously is only possible through abstract thinking. As in every T & L situation there is the teacher, the learner, the environment and the content transfer, the emphasis here is on the teacher and the content. With respect to the learner, volumes of research has been written about ‘how human beings learn’. With respect to the environment, there is the classroom environment or the virtual environment of an ODL situation and the institutional environment, where T & L takes place. Both must be conducive to teaching and learning. Tinto (1993) and Astin (1993) and many more since then shown that a non- conducive environment can lead to the learner dropping out of education. The teacher on the other hand, must be familiar with current teaching and learning theories and possess what Shulman (1986) called Pedagogical Content Knowledge (PCK).

Acquisition of knowledge begins with thinking while we are doing things or while we are doing things we think. The thinking could be concrete or abstract. When we are faced with a concept, such concept could be concrete or abstract. All misconceptions take place at this point. An abstract concept has to be understood by the learner. Skemp called it relational understanding compared to instrumental understanding, a result of surface learning. That is, it is concretised in the learner’s cognitive structure (Giannakopoulos, 1991). The formation of the concept is used as a scaffolding for the acquisition of a new related concept. Skemp (1977) explained that such examples are ‘misleading’ and can lead to misconceptions (as they do) since we deviate from the meaning of a variable. Skemp (1977: 27) argued that “...abstract concepts have their roots in manipulation of real objects but are never concretely attached to objects. They do have na analogous concrete character.”

Skemp (1977) used the word ‘abstraction’, a mental process used to ‘sieve’ through information (attributes) and chose the relevant ones. For Skemp (1977) abstraction was the beginning of the learning of any new concept. The use of abstraction becomes even more necessary when we have to learn secondary concepts which normally are abstract such as colour. Red, black and many others are primary concepts. We would not forget Skemp’s (1977:45) who said: If you want to see how good a teacher you are try to teach the word ‘red’ to a ...blind person. Never mind the word ‘colour’.

Skemp (1987) in trying to explain concept acquisition that leads to conceptual knowledge, he introduced a new idea which explains how concepts are acquired at various levels. He maintained that learning should be goal-directed, intelligent learning, and there is a continuous attempt to decrease the distance between the initial and final state so the two states coincide. Thus he developed a model for intelligence which comprises of two actions: Actions on environment or objects (Delta-1) and actions on formed concepts (Delta 2).

Figure 2. *A Model of Intelligence (adapted from Skemp, 1987: 57)*



In explaining the model (Figure 2) Skemp (1987) states that, in Delta 1, information is collected (by receptor) through sensory, conscious manner, and the mind acts on such objects, through abstraction. This way, a number of attributes of the object are identified and a mental image (object) is formed and passed on as information to Delta 2. As the mental image is processed intuitively it reflects on it and acts upon it and as a result the original information in Delta 1 is modified. This to and fro process continues till a permanent image (now a concept) is formed in Delta 2 as no more modification takes place in Delta 1. A number of interconnected concepts give rise to the formation of a schema. It is the learning of concepts through thinking and their relationships and principles that give rise to knowledge. Through initial abstractions, and categorisation of common features and initial image of the concept is formed. They are acquired through intuition. The image of the concept is stored in the conscious mind and is recalled consciously. Using further abstraction 'on abstractions', or reflecting on them the concept is attained in a higher level. Now the concept is stored in the subconscious. One then can operate on those concepts, using various procedures until these procedures are automated. A group of automated procedures gives rise to mental process. At that point, logical systems come into play.

Recognising the difference between primary (concrete) and secondary concepts (abstract) is a necessary condition for concept acquisition as different methods teaching methods have to be used. But concepts are not discrete entities. They are the smallest parts of the whole, like the atoms of a substance. Concepts are related to other concepts under some conditions. The concepts and their relationships give rise to principles. A principle is like a law that governs the conditions under which the concept will become part of the cognitive structure, if it fits in the existing structure (misfit implies it is a misconception) which can

be used to solve problems. It is what Piaget (1971) called assimilation. These principles form the structures of knowledge. Depending on their usage, they get the different names mentioned above, like declarative, procedural and so on. Use of these types of knowledge require certain skills (physical or mental). One of the most important skills according to Giannakopoulos (2012) is critical thinking. The complexity of critical thinking makes it impossible to define it and it is a concept that volumes have been written about it. However, it suffices here to say that logic, decision making, paying attention to detail, synthesis and analysis are just a few attributes (Halpern 2003). Briefly, Giannakopoulos (2012) found that to solve a problem it requires critical thinking. Critical thinking assist the learner to choose the appropriate knowledge and skills to apply it in the problem situation.

Problem Solving and Learning Theories

Problem solving is the cornerstone of school mathematics (National Council of Teachers of Mathematics [NCTM], 2000) and is a complex process that involves multiple variables (e.g. learner characteristics, task) (Xin, 2007). One critical factor in problem solving process relates to the characteristics of the problem solver, and therefore his or her behaviour to the task (e.g. the interaction between the problem solver and the task). Over the years many researchers have attempted to develop the ‘magic key’ that unlocks all problems. Progress has been made towards that but somehow not all learners can solve problems, be it in Mathematics or real life problems. Since problem solving has been accepted as a prerequisite to progress, volumes of research have been written about ‘how to solve a problem’ with as earlier stated Polya (1973) being the forefather of formulating a solution for a problem and was followed by others (Sternberg & William, 2002). What all these methods of solving problems have in common is that they all like a ‘recipe’. You have the ingredients (necessary knowledge), the tools (mental or physical) and you follow certain process (Steps 1 – 5, say) to the letter and ... problem solved. We all know even following this procedure to the letter somehow the problem is not solved. One of the reasons could be that problem solving is proceduralised. But problem solving is not just a process, as it is accepted by most researchers, but it is also a product (Giannakopoulos, 2012). The quality of the product plays a central role in problem solving. Problem solved? If yes, it satisfied the criteria of the quality. If not, then the quality of the solution was not acceptable, that is the desired product was not achieved.

It is possible that some teachers as the concentrate on procedures (procedural knowledge), they do not promote conceptual knowledge which can only be acquired by deep (not surface) learning, learning with understanding. What is interesting about these two types of knowledge is that they are *iterative*; the process is bidirectional. This means that the one feeds the other (Rittle-Johnson & Schneider, 2015). The better the procedural knowledge the better the conceptual knowledge. As a rule though, procedural knowledge should be the product of

conceptual knowledge. Once the procedure is perfected then it becomes automated as a result the cognitive load becomes less. As the learner becomes confident in using a procedure, he/she will become more creative with solutions, which will add to the conceptual knowledge. What must be kept in mind though is that when concepts are acquired and form part of the cognitive structure and easily retrieved, then they become declarative knowledge which is used to anchor a solution.

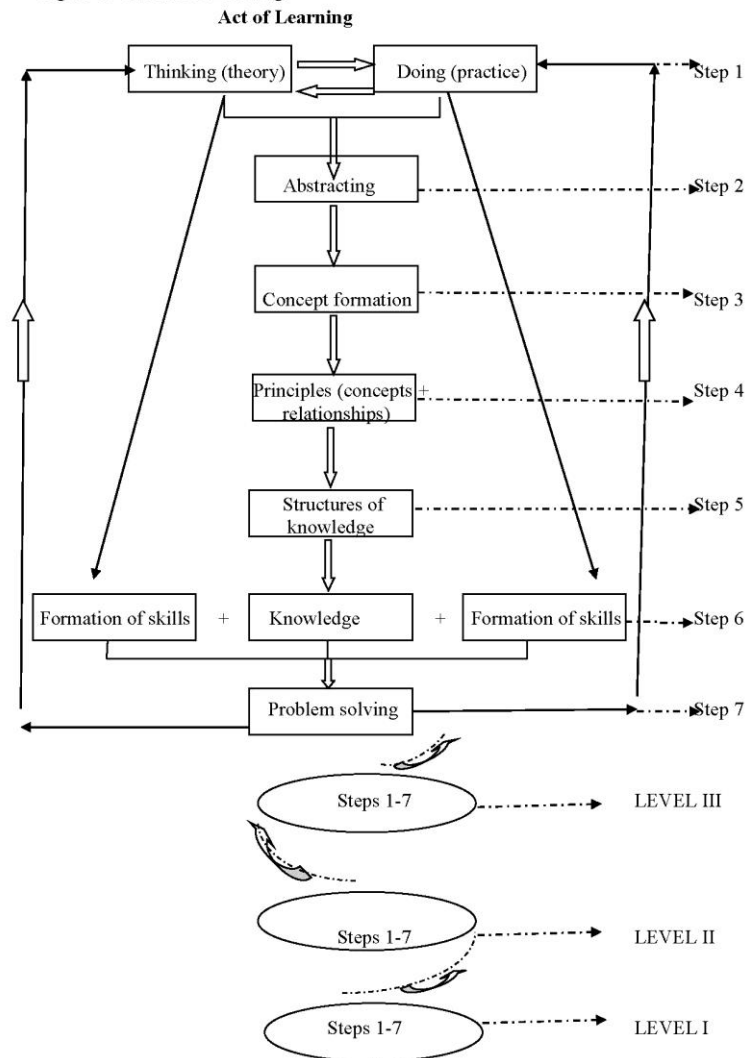
Problem Solving Model

The above exploration into Mathematics, its nature, knowledge acquisition, critical thinking and problem solving culminates into two important models (Giannakopoulos, 2012):

- a) The Act of learning model (see Figure 1)
- b) Problem solving conceptual model (see Figure 2).

The first model (Figure 1) can be used by Mathematics teachers in order to promote Mathematics as a living, applicable and useful subject and their teaching to promote understanding rather than instrumental (Skemp, 1977) or surface learning (Skemp, 1977; Rodd, 2002; Macleallan, 2005). Knowledge is constructed from thinking and doing. Using thinking we make abstractions about the concept to be learned. Once the concept is formed in the cognitive structure (assimilated and accommodated) it is combined with other related concepts which give rise to principles. These principles form the structures of knowledge and different types of knowledge. The acquired knowledge combined with various developed skills (mental or physical tools) assist us with the solution of problems. The connectivity between all prerequisites to acquisition of knowledge and of problem solving and critical thinking skills are highlighted. This connectivity is grounded in the ideas of Piaget (1971), Vygotsky (1978), Klausmeier (1979), Holmes (1985), Van Hiele (1986), and Resnick (1987). The “act of learning” is seen as a dual activity: thinking (theory) and doing (practice), i.e, praxis.

Figure 1: The Act of Learning.

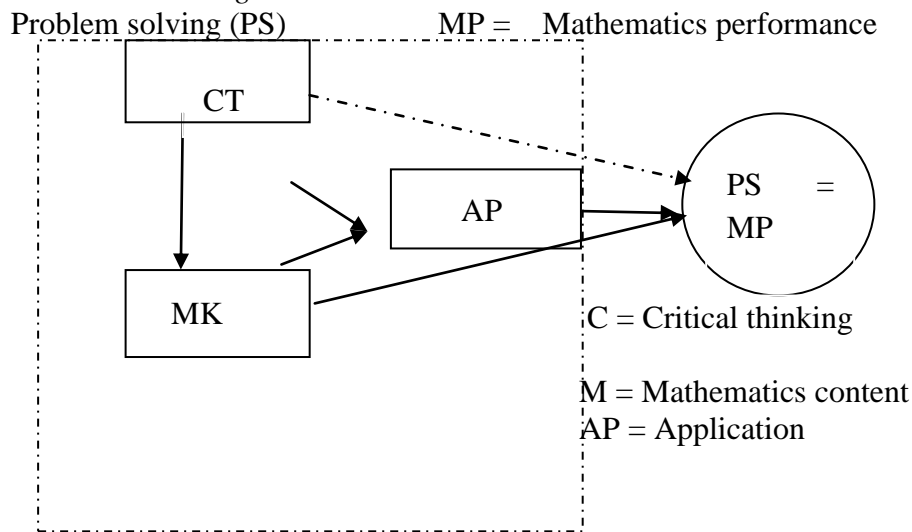


Source: Giannakopoulos (2012: 10)

Furthermore, a concept can be acquired in different levels, where steps 1-7 are repeated. For example if $3x+2 = 6$ (Level I), then $(x+1)(x+2) = x^2 + 6$ (could be Level III).

The problem solving conceptual problem appears in Figure 2.

Figure 2. Problem Solving as a Product

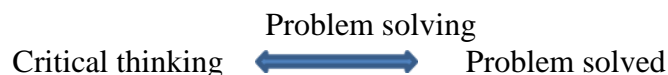


Source: Giannakopoulos (2012: 15)

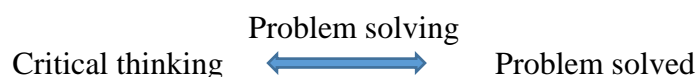
Using a holistic approach, problem solving and critical thinking are associated with learning because it is through thinking and abstracting that concepts are formed (Clements & Julie, 2004; Skemp, 1977) and subsequently knowledge. Viewing problem solving also as a product (problem solved, performance) creates a cyclical situation where critical thinking gives rise to problem solving and problem solving gives rise to higher levels of critical thinking. So problem solving (a means to an end, the process), once the problem is solved (the product) it is the beginning of solving another problem.

Conclusion

The above exploration on problem solving in general and Mathematics in particular, approaching it from existing learning theories perspective and problem solving theories two important shortfalls were identified: Problem solving is perceived only as a process and as a result emphasis is on procedures. The way knowledge is acquired is based mostly on transmission of knowledge and not on understanding. It was shown that Skemp's (1977) psychology of teaching and learning of mathematics in general and problem solving in particular, in a pragmatic way (praxis) based on deep (relational) understanding forms the basis of a solid knowledge base. Combining Skemp's idea in the 'Act of learning in Giannakopoulos (2012) problem solving model, where critical thinking plays a central role, then problem solving is viewed as a process as well as product where the more problems we solve the higher the levels of critical thinking are achieved. And the greater the use of critical thinking, knowledge acquisition is based on understanding which combined with other skills assist us to solve problems. Therefore:



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