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Applying Dynamical Discrete Systems to Teach Mathematics

Marta Peña

Athens Institute for Education and Research
8 Valaoritou Street, Kolonaki, 10683 Athens, Greece

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Marta Peña, Associate Professor, Polytechnic University of Catalonia, Spain

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ABSTRACT

The main goal is using examples of dynamical discrete systems in order to illustrate basic algebraic notions such as: the matrix of a linear map, their eigenvalues and eigenvectors. In particular: the computation of the matrix by means of consecutive states; and the eigenvectors and eigenvalues as stationary/ asymptotic distribution and growing; equilibrium points and its stabity; stochastic matrices. We include a guideline on discrete systems for faculty lacking in this topic.

Keywords: dynamical discrete systems, eigenvalues, eigenvectors, equilibrium point, stability.

Introduction

One of the recurrent controversies at our Engineering Faculty concerns the orientation of first-year basic courses, particularly the subject area of mathematics, considering its role as an essential tool in technological disciplines.

The Bologna process is a good opportunity to delve into this debate. In fact, the Faculty Council has decided that 20% of the credits from basic courses must be related to technological applications. Thus, a mathematical engineering seminar was held during an academic year at the Engineering Faculty of Barcelona, being the author one of the organizers. Sessions were each devoted to one technological discipline and aimed at identifying the most frequently used mathematical tools with the collaboration of guest speakers from mathematics and technology departments.

Moreover, the implementation of the Bologna process is presented as an excellent opportunity to substitute the traditional teaching-learning model by another one where students play a more active role. In this case, we can use the Problem-Based Learning (PBL) method. This environment is a really useful tool to increase student involvement as well as multidisciplinary. With PBL, before students increase their knowledge of the topic, they are given a real situation-based problem which will drive the learning process. Students will discover what they need to learn in order to solve the problem, either individually or in groups, using tools provided by the teacher or 'facilitator', or found by themselves. In contrast with conventional Lecture-Based Learning (LBL) methods, with the new technique students have more responsibility since they must decide how to approach their projects. They must identify their own learning requirements, find resources, analyze information obtained by research, and finally construct their own knowledge. Moreover, PBL helps students develop skills and competences such as group and self-assessment skills, which will allow them to keep up-to-date and continue to learn autonomously, or to become acquainted with the decision-making process, time scheduling and, last but not least, improve communication skills. Furthermore, theory and practice are integrated and motivation is enhanced, which results in increased academic performance.

A collection of exercises and problems was created to illustrate the applications identified in the seminar sessions. These exercises would be some of the real situation-based problems given for introducing the different mathematic topics. Two conditions were imposed: availability for first-year students and emphasis on the use of mathematical tools in technical subjects in later academic years. As additional material, guidelines for each technological area addressed to faculty without an engineering background were defined. Some of them have been already published by the author [1], [2].

The following exercises illustrate the use of Linear Algebra in different engineering areas. As said above, they are based on conclusions drawn at the above seminar.

- Set of complex numbers.
  Alternating current representation. Its use to calculate voltage drop and cancellation of reactive power.
- Matrices. Determinants. Rank.
  Controllability of Control Linear Systems. Observability of Control Linear Systems. Its controllability indices. Composition of Systems. Realizations.
- Linear System Equations.
  Network flows. Leontief economic model.
  Vector Spaces. Bases. Coordinates.
  Color codes. Crystallography. States in Discrete Systems. Control Functions in Discrete Systems.
  Vector Subspaces.
  Reachable states of Control Linear Systems. Circuit analysis (mesh currents, node voltages).
  Linear maps.
  Controllability and observability matrices. Kalman decomposition. Changes of Bases in the System Equations and Invariance of the Transfer Function Matrix.
  Diagonalization. Eigenvectors, eigenvalues.
  Strain and stress tensors. Circulant matrices. Dynamical Discrete Systems. Invariant Subspaces and Restriction to an Invariant Subspace. Controllable Subsystem. Poles and Pole Assignment.
  Non-diagonizable matrices;
  Control canonical form.
  Equilibrium points. Stability.
  Dynamical Discrete Systems. Dynamical Continuous Systems.
  Dynamical Discrete Systems.
  Leslie population model. Gould accessibility indices.

Here, we present some of this material: some exercises and the guideline for dynamical discrete linear systems. As general references regarding Linear Algebra, see for example [3], [4].

Guidelines for Dynamical Discrete Linear Systems

We include an example of guideline for faculty lacking an applied background.

Definition of Dynamical Discrete Linear Systems

Definition

(1) A *discrete linear system* with constant coefficients is an equation of the form

$$x(k+1) = Ax(k) + b(k)\,, \quad k \geq 0$$

where $A \in M_n$, $b(k) = (b_1(k),...,b_n(k))$ are n given discrete functions and $x = (x_1,...,x_n)$ are n discrete functions to determine such that they verify the above equality.

(1') In a more explicit form, if $A = (a_{ij})$:

$$x_1(k+1) = a_{11}x_1(k) + ... + a_{1n}x_n(k)$$
$$\cdots$$
$$x_n(k+1) = a_{n1}x_1(k) + ... + a_{nn}x_n(k)$$

(2) Then it is said that *x(k)*is a *solution* with *initial conditions* $x(0) = (x_1(0),...,x_n(0)) \in \mathbb{R}^n$.

(3) If *b(k)* = 0,the system is called *homogeneous* and, otherwise it is *complete*.

Resolution of Dynamical Discrete Linear Systems

Proposition (Homogeneous case)

We consider a homogeneous system:

$$x(k+1) = Ax(k)$$

(1) Fixed some initial conditions $x(0)$, there exists a unique solution of the system, given by:

$$\boxed{x(k) = A^k x(0)}, \qquad k \geq 0$$

(2) The set of solutions, when varying $x(0)$, forms a vectorial space S_0 of dimension *n*.

(2') A subset of solutions is linearly independent if and only if its initial conditions are so.

Observation

(1) In general, the computation of the powers A^k requires reducing the matrix A to its Jordan form A_J. Then

$$A_J = (S_J)^{-1} A S_J = \begin{pmatrix} J_1 & & \\ & J_2 & \\ & & \ddots \end{pmatrix} \Rightarrow x(k) = S_J A_J^k \, S_J^{-1} x_0 = S_J \begin{pmatrix} J_1^k & & \\ & J_2^k & \\ & & \ddots \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

We remind that $x(0) = S_J \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$

$$\begin{pmatrix} \lambda & & & \\ 1 & \lambda & & \\ & 1 & \lambda & \\ & & \ddots & \ddots \end{pmatrix}^k = \begin{pmatrix} \lambda^k & & \\ \binom{k}{1}\lambda^{k-1} & \ddots & \\ \binom{k}{2}\lambda^{k-2} & \ddots & \\ \vdots & \ddots & \end{pmatrix}$$

It is especially simple the diagonalible case, which we will see now. (2) If we denote by $(A^k)_1,\ldots,(A^k)_n$ the columns of A^k, we can write the general solution as

$$x(k) = (A^k)_1 x_1(0) + \ldots + (A^k)_n x_n(0)$$

which shows that the columns of A^k form a basis of S_0.

<u>Corollary</u> (Diagonalizable Homogeneous Case) – In the Above Conditions:

(1) If $x(0) \equiv v$ is an eigenvector of A with eigenvalue $\lambda \in \mathbb{R}$, its corresponding solution is:

$$x(k) = \lambda^k v$$

(2) If A diagonalizes, and$v_1,...,v_n$ is a basis of eigenvectors, with respective eigenvalues $\lambda_1,...,\lambda_n$, all of them real, any solution is of the form:

$$\boxed{x(k) = (v_1 \ldots v_n)\begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{pmatrix}\begin{pmatrix} c_1 \\ \ldots \\ c_n \end{pmatrix} = c_1\lambda_1^k v_1 + \ldots + c_n\lambda_n^k v_n}$$

where the coefficients $c_1,...,c_n$ are determined by the initial conditions $x(0)$:

$$x(0) = c_1 v_1 + \ldots + c_n v_n$$

that is, they are the coordinates of $x(0)$ in the basis of eigenvalues.

<u>Definition</u>

(1) The solutions of the above form

$$x(k) = \lambda^k v,\ k \in \mathbb{N}$$

where v is an eigenvector, of eigenvalue $\lambda \in \mathbb{R}$, are called *proper modes* of the system.

(2) If λ_1 is a real and simple eigenvalue, and its modulus is greater than the remainder eigenvalues

$$|\lambda_1| > |\lambda_2|, |\lambda_3|, \ldots$$

it is called *dominant eigenvalue* and the corresponding proper mode

$$x(k) = \lambda_1^k v_1$$

dominant mode. For any other solution of the form

$$x(k) = c_1 \lambda_1^k v_1 + \ldots,\ c_1 \neq 0$$

this first addend is called its *dominant part*.

(2') Analogously, if λ_2 is a simple eigenvalue and

$$|\lambda_1| > |\lambda_2| > |\lambda_3|, |\lambda_4|, \ldots$$

it is called *subdominant eigenvalue* and the corresponding proper mode *subdominant mode*.

Observation

The above corollary generalizes to the case of conjugate complex eigenvalues. We suppose, for example, $\lambda_2 = \overline{\lambda_1}$. Then, we can take $v_2 = \overline{v_1}$, and it must be $c_2 = \overline{c_1}$ so that it has real coordinates.

Proposition (Complete Case)

Given a complete system

$$x(k+1) = Ax(k) + b(k)$$

the set of solutions is

$$\boxed{S = x_0(k) + S_0}$$

where:

- S_0 is the set of solutions of the associated *homogeneous system*, $x(k+1) = Ax(k)$.
- $x_0(k)$ is any *particular solution*; for example, the one corresponding to $x(0) = 0$:

$$x_0(k) = A^{k-1}b(0) + A^{k-2}b(1) + \ldots + b(k-1)$$

Dynamical Behaviour of the Proper Modes

First we see that the proper modes are invariant (or stationary) solutions.

Definition

(1) Given an homogeneous system

$$x(k+1) = Ax(k)$$

A subspace $F \subset \Re^n$ is called (dynamically) *invariant* if any other solution which begins in *F* it is maintained inside of *F*:

$$x(0) \in F \Rightarrow x(k) \in F, \forall k \in \mathbb{N}$$

It is obvious that it is equivalent to the "algebraically invariant" condition:

$$A(F) \subset F$$

(2) The following particular cases are especially interesting:

(2.1) It is called *of output* (or escaping) if any other (non null) solution in F is not bounded.
(2.2) It is called *of input* if any other solution in F converges to the origin.

We analyze now that the one-dimensional invariant subspaces are spanned by real eigenvectors of A and they correspond to real proper modes of the system. The two-dimensional case corresponds to conjugate complex eigenvalues, as we will see later.

<u>Proposition</u> (Behaviour of the Real Proper Modes)

Given an homogeneous system

$$x(k+1) = Ax(k)$$

(1) If $v \in \mathbb{R}^n$ is an eigenvector (and only in this case) it is verified that:

$$x(0) \in F \equiv [v] \Rightarrow x(k) \in F, \forall k \in \mathbb{N}$$

(1') In a more precise form, we remind that:

$$x(0) = cv \Rightarrow x(k) = c\lambda^k v$$

where $\lambda \in \mathbb{R}$ is the eigenvalue of v.

(2) Then,

(2.1) If $|\lambda| > 1$, $F \equiv [v]$ is "of output", that is:

$\{x(k), k \in N\}$ not bounded, $\forall x(0) \in F, x(0) \neq 0$.

(2.2) If $|\lambda| < 1$, $F \equiv [v]$ is a straight line "of input", that is:

$x(k) \to 0, \ \forall x(0) \in F$.

(2.3) If $|\lambda| = 1$, $F \equiv [v]$ is a straight line of "fixed points", that is:

$x(k) = 0, \ \forall k, \forall x(0) \in F$.

(2.4) If $|\lambda| = -1$, all the solutions in $F \equiv [v]$ are oscillating, that is:

$x(k) = (-1)^k x(0), \ \forall k, \forall x(0) \in F$.

<u>Observation</u>

(1) In other words, the proper modes are the only solutions which keep the proportion between the different coordinates $x_1(k),...,x_n(k)$. In this sense, they are called *stationary modes*. This stationary distribution is given by the coordinates of the corresponding eigenvector.

(1') Then the eigenvalue is the *growth rate*, the same for all the coordinates, and therefore the global for the set of population.

(2) We will see that generically the solutions converge to the dominant mode. So, the coordinates of the dominant eigenvector give the *a symptotic stationary distribution* and the eigenvalue, the *asymptotic growth rate*.

As we have said, the eigenvectors of conjugate complex eigenvalues span two-dimensional invariant subspaces, with rotatory behaviours:

Observation (Conjugate Eigenvalues: Rotatory Modes)

Given an homogeneous system

$$x(k+1) = Ax(k)$$

And we assume conjugate complex eigenvalues and their corresponding eigenvectors:

$$\lambda_1 = |\lambda_1| e^{i\omega}, \lambda_2 = \overline{\lambda_1} = |\lambda_1| e^{-i\omega}$$

$$v_1 = u + iw, v_2 = \overline{v}_1 = u - iw$$

(1) The plane F is invariant:

$$x(0) \in F \Rightarrow x(k) \in F, \forall k \in \mathbb{N}$$

(1') More precisely. In the basis (u, w): $\hat{x}(k) = |\lambda_1|^k e^{i\omega k} \hat{x}(0)$. So:

(1'.1) If $|\lambda_1| > 1$, F is an escaping plane (in fact, the solutions are divergent spirals).

(1'.2) If $|\lambda_1| < 1$, F is an entry plane (in fact, the solutions are convergent spirals to the origin).

(1'.3) If $|\lambda_1| = 1$, F is a plane of turns.

(2) If λ_1 is simple, and $|\lambda_1| > |\lambda_3|, |\lambda_4|, \ldots$, the above solutions are dominant, and the remainder ones converge asymptotically to them.

Asymptotic Convergence to the Dominant Mode

Proposition (Asymptotic Convergence to the Dominant Mode)

Given an homogeneous discrete system

$$x(k+1) = Ax(k)$$

with A diagonalizable. Being $\lambda_1, \ldots, \lambda_n$ the eigenvalues, and $(v_1, \ldots, v_n)$ a basis of eigenvectors.

We suppose λ_1 is a dominant eigenvalue and $x(k)$ is a solution, with initial conditions

$$x(0) = c_1 v_1 + \ldots + c_n v_n$$

Then:

(1) $\boxed{x(k) \cong c_1 \lambda_1^k v_1}$, for $k >>$, if $c_1 \neq 0$.

More precisely:

$$\lim_k \frac{x(k)}{\lambda_1^k} = c_1 v_1, \qquad \text{if } c_1 \neq 0$$

(1') In particular, if $c_1 \neq 0$:

$$\lim_k \frac{\|x(k+1)\|}{\|x(k)\|} = |\lambda_1|$$

$$\lim_k \frac{x(k)}{\|x(k)\|} = \mathrm{sgn}(c_1)\frac{v_1}{\|v_1\|}$$

(2) Moreover, if λ_2 is a subdominant eigenvalue, then:

$$\lim_k \frac{\|x(k) - c_1\lambda_1^k v_1\|}{\|x(k-1) - c_1\lambda_1^{k-1} v_1\|} = |\lambda_2|, \text{ if } c_2 \neq 0$$

Observation – We can say, then, that *x(k)* asymptotically converges to its dominant part $c_1\lambda_1^k v_1$, with velocity of approximation $|\lambda_2|$.

An important application of the above proposition is the following corollary, basis of a lot of numerical algorithms for the computation of eigenvalues and eigenvectors. The hypothesis of existence of dominant eigenvalue can be guaranteed, for example, by means of the Perron-Frobenius theorems for matrices with positive coefficients, as we will see. Then, moreover, the dominant eigenvalue results positive and real (so, $|\lambda_1| = \lambda_1$).

Corollary (power method for the computation of the dominant eigenvalue and the dominant eigenvector)

We suppose that a diagonalizable matrix *A* has a dominant eigenvalue λ_1. Then:

$$\lim_k \frac{\|A^{k+1}w\|}{\|A^k w\|} = |\lambda_1|$$

$$\lim_k \frac{A^k w}{\|A^k w\|} \text{ is a dominant eigenvector.}$$

for $w \in \mathbb{R}^n$ generic (more precisely: that in a basis of eigenvectors have the first coordinates non null).

Equilibrium Points. Stability

Finally, we study the behaviour of the solutions with regard to the equilibrium point.

Definition – We consider a discrete system of the form

$$x(k+1) = Ax(k) + b$$

(1) A constant solution x_c is called an *equilibrium point*

$$x_e = Ax_e + b$$

(2) We assume a unique equilibrium point x_e. It is called:

(2.1) *Unstable* if some other solution is not bounded.

(2.2) *Asymptotically stable* if any other solution converges to x_e.

$$\lim_k x(k) = x_e$$

(2.3) *Marginally stable* if any other solution is bounded, but there is some solution not convergent to x_e.

Proposition (Stability of an Equilibrium Point)

We consider a discrete system of the form

$$x(k+1) = Ax(k) + b$$

(1) There exists a unique equilibrium point x_e if and only if 1is not eigenvalue of *A*, and then

$$x_e = (I - A)^{-1} b$$

(2) Then:

(2.1) If $|\lambda| > 1$ for some eigenvalue of *A*, it is unstable.

(2.2) If $|\lambda| < 1$ for all the eigenvalues of *A*, it is asymptotically stable.

(2.3) If $|\lambda| \le 1$ for all the eigenvalues of *A*, the system is marginally stable if and only if the eigenvalues with $|\lambda| = 1$ have the same geometric multiplicity than algebraic multiplicity; otherwise, it is unstable.

(2') In particular, if there is dominant eigenvalue λ_1:

(2'.1) $|\lambda_1| > 1 \Rightarrow$ unstable.

(2'.2) $|\lambda_1| < 1 \Rightarrow$ asymptotically stable.

(2'.3) $|\lambda_1| = 1 \Rightarrow$ marginally stable.

Observation – Analogously to (2') above, if λ_1 is a complex and simple eigenvalue, with $\lambda_2 = \overline{\lambda_1}$ and $|\lambda_1| > |\lambda_3|, |\lambda_4|, \ldots$:

(1) $\lambda_1 > 1 \Rightarrow$ unstable.

(2) $\lambda_1 < 1 \Rightarrow$ asymptotically stable.

(3) $\lambda_1 = 1 \Rightarrow$ marginally stable.

In fact, the dominant solutions are turns around x_e, and the other generic solutions tend asymptotically to them.

Positive Matrices

We have seen that, in the dynamical behaviour of a discrete system, the existence of a dominant eigenvalue plays an important role. We will see that this is guaranteed for matrices with positive coefficients:

Theorem (Perron, 1907)

We consider a discrete system

$$x(k+1) = Ax(k)$$

with A a positive matrix, that is:

$$A = (a_{ij}), a_{ij} > 0 \qquad \forall 1 \le i, j \le n$$

Then:
(1) There exists a dominant eigenvalue λ_1 (so, real and simple) and $\lambda_1 > 0$.
(2) Its (normalized) eigenvector also has positive coordinates.
(3) It is the only (normalized) eigenvector with non-negative coordinates.

Observation

(1) The dominant eigenvalue λ_1 is usually called “Perron root”, and it verifies:

$$\min_i \sum_j a_{ij} \le \lambda_1 \le \max_i \sum_j a_{ij}$$

(1’) As a Perron eigenvector it is usually taken the one that has sum of coordinates equal to 1 (usually called “stochastic” eigenvector).
(2) The condition $A > 0$ is frequent in population models, where the variables must be positive. Then:

- The Perron eigenvalue gives the asymptotic growth rate.
- The coordinates of the stochastic eigenvalue give the asymptotic population distribution.

(3) The above theorem is applied in particular to stochastic matrices ($\Leftrightarrow$ the sum of the coefficients of each columns is 1). Then the Perron eigenvalue is 1. Moreover it can be ensured that there exists

$$A_\infty = \lim_k A^k$$

and that it is a stochastic positive matrix of rank 1. In fact all the columns are equal to the stochastic eigenvector.
(3’) Stochastic matrices appear just when

$$x_1(k) + \cdots + x_n(k) = \text{constant}$$

For example, when the global population is constant.
(4) The Frobenius theorem (1912) generalizes the Perron theorem to a wider family of matrices $A \ge 0$, the so-called “primitive” matrices.

The Particular Case of Equations in Linear Finite Differences

Definition

(1) We will denote by $y(k)$, $k \in \mathbb{N}$ *a discrete variable function* (or, simply, "discrete function"), that is, y: $\mathbb{N} \to \mathbb{R}$. It can be identified with the sequence ($y(0)$, $y(1)$,…,$y(k)$,…).
(2) An *equation in linear finite differences* with constant coefficients of *order n* is an equation of the form:

$$y(k+n)+a_{n-1}y(k+n-1)+...+ay(k+1)+a_0y(k)=\varphi(k) \qquad (*)$$

where a_j are constants $0 \le j \le n-1$, φ is a given discrete function and y is a discrete function to determine such that it verifies the above relation (*).
(2') Then it is said that *y(k)* is a *solution* of (*), with *initial conditions* $y(0)$,…, $y(n\text{-}1)$.
(2'') If $\varphi = 0$, the equation in differences is called *homogeneous*. Otherwise it is called *complete,* and the *associated homogeneous* equation is the one that results of substituting $\varphi(k)$ by 0.
(3) It is called *characteristic polynomial* of (*) the following one:

$$Q(\tau) = \tau^n + a_{n-1}\tau_{n-1} + ... + a_1\tau + a_0$$

Its roots are called *characteristic values.*
(3') If there is one root which is real and simple with modulus strictly greater than the remainder ones, it is called *dominant* (and the following one, if there exist, *subdominant*).

Observation

The equation (*) indicates the way of constructing the sequence $y(k)$ *by recurrence*, and the first objective is to find an *explicit expression* of the general term $y(k)$.

Proposition

An equation in linear finite differences

$$y(k+n)+a_{n-1}y(k+n-1)+...+ay(k+1)+a_0y(k)=\varphi(k)$$

can be considered a particular case of discrete linear system. Indeed, if we define $x_1,\ldots,x_n$ by:

$$x_1(k) = y(k), x_2(k) = y(k+1),..., x_n(k) = y(k+n-1)$$

it results

$$x(k+1) = Ax(k) + b(k)$$

$$A=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix}, b(k)=\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \varphi(k) \end{pmatrix}$$

So, the first coordinate $x_1(k)$ of the solution $x(k)$ of this system is the solution $y(k)$ of the initial equation in linear finite differences.

We observe that A is a "companion" matrix, with characteristic polynomial the one of the output equation in linear finite differences. So, the eigenvalues of A are the characteristic values of the equation in linear finite differences, with the same multiplicities.

Corollary

Given a homogeneous equation in linear finite differences:

(1) If the characteristic values are $\lambda_1,...,\lambda_n \in \mathbb{R}$, all different (which it is equivalent to say that all the characteristic values are simple), then the general solution is of the following form:

$$y(k) = c_1\lambda_1^k + ... + c_n\lambda_n^k$$

where $c_1,\ldots,c_n$ are determined by the initial conditions $y(0),\ldots, y(n\text{-}1)$, that is:

$$y(0) = c_1 + ... + c_n$$
$$y(1) = c_1\lambda_1 + ... + c_n\lambda_n$$
$$\cdots$$
$$y(n-1) = c_1\lambda_1^{n-1} + ... + c_n\lambda_n^{n-1}$$

More explicitly,

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 1 & \cdots & \cdots & 1 \\ \lambda_1 & \cdots & \cdots & \lambda_n \\ \cdots & & & \cdots \\ \lambda_1^{n-1} & & & \lambda_n^{n-1} \end{pmatrix}^{-1} \begin{pmatrix} y(0) \\ y(1) \\ \vdots \\ y(n-1) \end{pmatrix}$$

where the matrix is effectively invertible if $\lambda_1,...,\lambda_n$ are different (Vandermonde determinant).

(2) In particular, if we suppose that there exists a dominant characteristic value λ_1. Then:

$$\lim_k \frac{y(k+1)}{y(k)} = \lambda_1, \qquad \text{if } c_1 \neq 0.$$

If, moreover, λ_2 is the subdominant characteristic value:

$$\lim_k \frac{y(k) - c_1\lambda_1^k}{c_2\lambda_2^k} = \lambda_2, \text{ if } c_2 \neq 0.$$

Exercises in Dynamical Discrete Linear Systems

We now present some exercises in dynamical discrete linear systems.

Exercise 1 (Emigration)

We consider the population exchange city center/suburbs, assuming to simplify that the total number of habitants is constant, equal to 1.000.000. We denote by x_1, x_2 the census in the city center and the suburbs, respectively, in thousands of habitants.

(a) If the value of x_1 the years 1995, 2000 and 2005 was 600, 400 and 300, respectively, compute the prevision for 2010, assuming that the linear transformation which modelizes the five-year change of (x_1, x_2) is the same in all the period.
(b) Determine the matrix *A* of this linear map.
(c) Think on which conditions the city center will be empty and on the contrary, when it will reach an equilibrium point. In this case, compute this constant census and analyze if in any conditions one converges to it.

Exercise 2 (Dam/Predator)

(A) We consider the simplified model of dam/predator

$$\begin{aligned} D_{k+1} &= 0'5D_k + 0'4P_k \\ P_{k+1} &= -0'125D_k + 1'1P_k \end{aligned}$$

where P_k, D_k indicate the number of dams and predators, respectively, the year *k*. We can state it in the form of a discrete system:

$$x(k+1) = Ax(k); \quad x(k) = \begin{pmatrix} D_k \\ P_k \end{pmatrix}, \quad A = \begin{pmatrix} 0'5 & 0'4 \\ -0'125 & 1'1 \end{pmatrix}$$

(a) Determine the eigenvalues and eigenvectors of the matrix A.
(b) Find the solution of this model, depending on the initial conditions. Explain the result.

(B) Now we consider the model of dam/predator depending on α :

$$x(k+1) = A_\alpha x(k); \quad x(k) = \begin{pmatrix} D_k \\ P_k \end{pmatrix}, \quad A_\alpha = \begin{pmatrix} 0'5 & 0'4 \\ -\alpha & 1'1 \end{pmatrix}$$

where α represents the voracity of the predators.

(a) Prove that the eigenvalues of the matrix A_α are lower than 1 if and only if $\alpha > 0'125$.Justify that then both species converge to extinction.
(b) Justify analogously that for $\alpha < 0'125$ the populations converge to increase and that for $\alpha = 0'125$, to stabilize.
(c) Determine the distribution of the population which converges for $\alpha = 0'125$and $\alpha = 0'104$. Explain the result.

Exercise 3(American Owl)

In the study of Lamberson (1992) about the survival of the American owl, he experimentally obtained that:

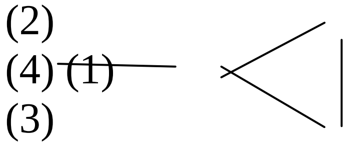
$$\begin{pmatrix} Y_{k+1} \\ S_{k+1} \\ D_{k+1} \end{pmatrix} = A \begin{pmatrix} Y_k \\ S_k \\ D_k \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 0'33 \\ 0'18 & 0 & 0 \\ 0 & 0'71 & 0'94 \end{pmatrix}$$

where Y_k, S_k and D_k indicates the "young" population (until 1 year old), "subadult" population (between 1 and 2 years old) and "adult" population, respectively, the year *k*.

(a) Explain the coefficients of matrix *A*.
(b) Verify that the eigenvalues of matrix *A* are approximately: 0´98, $-0'02 \pm 0'21i$.
(c) Deduce that in these conditions, the American owl converges to extinction.
(d) Verify that an improvement of the survival in the transition sub adult/adult does not avoid extinction.
(e) In contrast, verify that extinction would be avoided if the young survival index is 30% instead of 18%.
(Remark: the referred survival index was effectively improved by means of appropriate forest policies.)
(f) In these conditions, compute the year increase index of the global population, and the population distribution between young, sub adult and adult which converge.

Exercise 4(Gould Accessibility Indices)

The "Gould accessibility indices" have been used in some geographic problems, as for example transport nets or migration movements. For their determination a net (or graph) is configured representing the cities (or other entities geographically significant) and the connections between them. For example:



Assuming that the knots or vertices are numbered (*i=1,…,n*), it is called its (modified) adjacency matrix the symmetric matrix $A=(a_{ij})\in M_n(R)$ defined by: $a_{ii}=1$; $a_{ij}=1$ or 0, depending if the corresponding vertices are or not connected by an arc. So, for the above graph it would be:

$$A=\begin{pmatrix}1&1&1&1\\1&1&1&0\\1&1&1&0\\1&0&0&1\end{pmatrix}$$

Thus, for the system $x(k+1)=Ax(k)$, the relation between the situations at the instants *k* and *k*+1 is given by:

$$\begin{aligned}x_1(k+1)&=x_1(k)+x_2(k)+x_3(k)+x_4(k)\\x_2(k+1)&=x_1(k)+x_2(k)+x_3(k)\\x_3(k+1)&=x_1(k)+x_2(k)+x_3(k)\\x_4(k+1)&=x_1(k)+x_4(k)\end{aligned}$$

More in general, we suppose that initially in each vertex there is a certain number (not null) of objects $x_1(0),...,x_n(0)$, and that in each unity of time each object creates a copy in each one of the adjacent vertices, being $x_1(k),...,x_n(k)$ the objects which are in the corresponding vertex, at the instant *k*. For example, possible sequences in the above example would be:

(1,1,1,1),(4,3,3,2),(12,10,10,6),(38,32,32,18),…
(1,2,3,4),(10,6,6,5),(27,22,22,15),(86,71,71,42),…
(4,3,2,1),(10,9,9,5),(33,28,28,15),(104,89,89,48),…

It can be proved that its greater eigenvalue is positive and simple, and that we can take as a basis of its 1-dimensional subspace associated to this eigenvalue a vector with positive coordinates, whose sum is 1. These coordinates are called the Gould accessibility indices of the corresponding vertices.

(a) Determine the Gould accessibility indices for the vertices of the above graph.
(b) Analyze if the obtained result fits with the intuitive idea of "accessibility" of each vertex.
(c) Prove that, asymptotically; the percentage of objects in each vertex is given by the Gould accessibility indices, independently of the initial situation.

(d) Determine, in the above sequences, the percentage of objects in each vertex for k=3 and compare them with the obtained indices in (a).

Exercise 5 (Bus Stations)

Let us consider four bus stations A, B, C, D. The traffic is determined by the following rules:

(i) Stations A, B: 1/3 of buses go to C; 1/3 of buses goes to D; 1/3 of buses remains for maintenance.
(ii) Station C (respectively D): 1/4 of buses goes to A; 1/4 of buses goes to B; 1/2 of buses goes to D (respectively C).

Prove that there is asymptotic stationary distribution of the buses, and compute it.

Exercise 6 (Engineering School)

(A) In a very demanding Engineering School there only approve the 30% of the students each course of the degree. The remainder students repeat the course, except in first course, where the 50% of the students leave the degree. Every year 600 new students enter in the school. Then:

(a) Determine the equations which govern the number of students every course.
(b) Compute the equilibrium point and study its stability.

(B) Considering the excessive massification, a very drastic change in the pedagogical system inverted the percentages, in such a way that the percentage of students that approve every course was 70%. These better expectations reduce the abandonments at first course up to 10%. Then:

(a) Determine the new equations which govern the number of students every course.
(b) Compute the new equilibrium point and study its stability.

Exercise 7 (Fibonacci Numbers)

The Fibonacci numbers are:

$$1,1,2,3,5,8,13,21,\ldots$$

It is probably the first recurrent equation known in history which appears at the "Liber Abaci" from 1202, collecting a problem of rabbit breeding, stated and solved by Leonardo of Pisa.

These numbers are given by the Fibonacci equation, which is an equation in linear finite differences:

$$y(k+2) = y(k+1) + y(k)$$

with initial conditions:

$$y(0) = 1, y(1) = 1$$

Then:

(a) Determine the solution of this equation.

(b) Compute $\lim_{k} \frac{y(k+1)}{y(k)}$.

Solution of Exercises

Solution of Exercise 1 (Emigration)

(a) Being

$$x(0) = \begin{pmatrix} 600 \\ 400 \end{pmatrix}, \quad x(1) = \begin{pmatrix} 400 \\ 600 \end{pmatrix}, \quad x(2) = \begin{pmatrix} 300 \\ 700 \end{pmatrix}$$

Then:

$$x(2) = -\frac{1}{2}x(0) + \frac{3}{2}x(1)$$

Hence

$$x(3) = -\frac{1}{2}x(1) + \frac{3}{2}x(2) = \begin{pmatrix} 250 \\ 750 \end{pmatrix}$$

(b) Acting as in (a) for the vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we obtain the columns of:

$$A = \begin{pmatrix} 0'6 & 0'1 \\ 0'4 & 0'9 \end{pmatrix},$$

which is a stochastic matrix.

(c) $\begin{pmatrix} x_1^e \\ x_2^e \end{pmatrix} = \begin{pmatrix} 0'6 & 0'1 \\ 0'4 & 0'9 \end{pmatrix} \begin{pmatrix} x_1^e \\ x_2^e \end{pmatrix}$ if and only if $x_1^e = 200$.

It corresponds to the Perron eigenvalue 1, so that it is the asymptotic distribution.

Solution of Exercise 2 (Dam/Predator)

(A)

(a) The eigenvalues and eigenvectors of *A* are

$$\lambda_1 = 1 \qquad v_1 = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$\lambda_2 = 0'6 \qquad v_2 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

The first one indicates a stationary distribution of 4 predators for each 5 dams, which maintains the total population constant $(\lambda_1 = 1)$.

The second one indicates another stationary distribution (4 predators for each dam), with a yearly decrease of the total population of 40% $(\lambda_2 = 0'6)$. It fits with the appreciation that an excess of predators provoques a decrease of dams, and as a consequence of predators as well.

We will see now that, being $\lambda_1 = 1$ dominant, we will converge generically to the first situation.

(b) The solutions are of the form

$$x(k) = c_1 \lambda_1^k v_1 + c_2 \lambda_2^k v_2$$

with $\lambda_1 = 1$ dominant eigenvalue. The convergence to the dominant mode, if $c_1 \neq 0$, is clear so that:

$$c_2 \lambda_2^k v_2 = c_2 \left(\frac{1}{2}\right)^k v_2 \to 0$$

The condition $c_1 \neq 0$ depends on the initial conditions:

$$x(0) = c_1 v_1 + c_2 v_2$$

(b.1) $c_1 = 0$ only when the initial population distribution is 4/1 of v_2. Then $x(k)$ mantains this distribution, with a yearly decrease of 40% for both species, converging to extinction.

(b.2) For any other initial distribution is $c_1 \neq 0$, and then $x(k)$ converges to the dominant mode. But we have to distinguish the cases $c_1 > 0$ and $c_1 < 0$, which correspond to the initial proportion of predators being lower and greater, respectively, to the above 4/1:

$$\frac{4}{1} > \frac{x_1(0)}{x_2(0)} = \frac{4c_1 + 4c_2}{5c_1 + c_2} \Leftrightarrow 20c_1 + 4c_2 > 4c_1 + 4c_2 \Leftrightarrow c_1 > 0$$

When the initial proportion of predators is lower tan 4/1, the solution converges to the stabilization of the total population, with an asymptotic population distribution 4/5.

When the initial proportion of predators is greater than 4/1, then $c_1 < 0$ and the solution converges to negative coordinates, which means the extinction of both species in short term. For example, for $x(0) = (20,1)$ results $c_1 = -1, c_2 = 6$ and $x(3)$ has negative coordinates.

(B)

(a) Now the dominant eigenvalue is

$$\lambda_1 = 0'8 + \sqrt{0'9 - 0'4\alpha}$$

The dominant mode implies the stabilization of the populations when $\lambda_1 = 1$, which corresponds to the value $\alpha = 0'125$ above.

For greater voracities, the dominant eigenvalue is lower than 1, and the populations converge to extinction for any initial conditions.

(b) For lower voracities, the dominant mode supposes the increase of the populations, with a bigger proportion of dams than in the case $\alpha = 0'125$.

(c) For example, for $\alpha = 0'104$ is $\lambda_1 = 1'02$ and $v_1 = (10,13)$: the populations yearly increase 2%, and the proportion dams/predators is 1’3 (instead of 1’25 when $\alpha = 0'125$).

Solution of Exercise 3 (American Owl)

(a) The first row is formed by the birth rate. So, the young and sub adult population does not procreate, while each adult couple has, on average, 2 children each 3 years.

The coefficients 0´18 and 0´81 are the survival indices of the transition young/sub adult and sub adult/adult, respectively. It is clearly confirmed that the first one is critical: when the young phase finishes, they have to leave the nest, find a hunting domain, find a couple, construct a nest, etc.

The coefficient 0´94 indicates that the adult population has a yearly death rate of 6%.

(b) Direct computation. In particular we observe that tr*A* and det*A* are, respectively, the sum and the product of the stated eigenvalues.

(c) We have seen that any solution converges to the origin when $|\lambda| < 1$ for all eigenvalues.

(d) The extinction is avoided if and only the dominant eigenvalue is greater than 1: $\lambda_{DOM} \geq 1$

We have seen that:

$$a_{32} = 0'71 \Rightarrow \lambda_{DOM} = 0'98$$

If we increase this survival index, the dominant eigenvalue will increase, but it will not become 1:

$$a_{32} = 0'85 \Rightarrow \lambda_{DOM} = 0'9914$$

$$a_{32} = 0'94 \Rightarrow \lambda_{DOM} = 0'9963$$

$$a_{32} = 1'00 \Rightarrow \lambda_{DOM} = 0'9995$$

(e) For $a_{21} = 0'30$, instead of 0´18, the eigenvalues become 1´01, $-0'03 \pm 0'26i$. So:

$$\lambda_{DOM} = 1'01 > 1$$

(f) The asymptotic increase is given by the dominant eigenvalue, so it would yearly converge to 1%, for each cohort of population.
The asymptotic population distribution between the 3 cohorts is given by the coordinates of the eigenvector corresponding to the dominant eigenvalue:

$$v_{DOM} \approx (10,3,21).$$

That is, for each 10 young owls, there will be 3 sub adult owls and 31 adult owls, with a growth rate of 1%.

Solution of Exercise 4 (Gould Accessibility Indices)

(a) The eigenvalues and eigenvectors of matrix *A* are:

$$\lambda_1 = 3'17 \qquad v_1 = (0'61,0'52,0'52,0'28)$$
$$\lambda_2 = 1'31 \qquad v_2 = (0'25,-0'37,-0'37,0'82)$$
$$\lambda_3 = 0 \qquad v_3 = (0,0'71,-0'71,0)$$
$$\lambda_4 = -0'48 \qquad v_4 = (0'75,-0'30,-0'30,-0'51)$$

The Gould accessibility indices would be given by the coordinates of the dominant eigenvalue, v_1, normalized to sum 1:

$$\overline{v_1} = (0'32,0'27,0'27,0'14)$$

(b) It matches that the vertex 1 is the best connected and the 4, the worst. Equally that the 2 and 3 have the same index.
It is less intuitive that the index of 2 and 3 is almost the double than the one of 4, and only a 20% below than the one of 1.

(c) Asymptotically, the objects converge to be distributed according to the coordinates of the dominant eigenvalue, v_1.

(d) If we normalize (to sum 1) *x*(3) for the different initial conditions, we obtain:

$$x(0) = (1,1,1,1), \qquad \overline{x}(3) = (0'31,0'27,0'27,0'15)$$
$$x(0) = (1,2,3,4), \qquad \overline{x}(3) = (0'32,0'26,0'26,0'16)$$
$$x(0) = (4,3,2,1), \qquad \overline{x}(3) = (0'31,0'27,0'27,0'14)$$

which is very similar to the Gould indices $\overline{v_1}$.

Solution of Exercise 5 (Bus Stations)

Clearly the total number of buses is constant, so that the matrix of the discrete system will be stochastic. Indeed, the number of buses $x_A(k)$, $x_B(k)$, $x_C(k)$ and $x_D(k)$ in the respective station at the day *k*, is determined by

$$\begin{pmatrix} x_A(k+1) \\ x_B(k+1) \\ x_C(k+1) \\ x_D(k+1) \end{pmatrix} = \begin{pmatrix} 1/3 & 0 & 1/4 & 1/4 \\ 0 & 1/3 & 1/4 & 1/4 \\ 1/3 & 1/3 & 0 & 1/2 \\ 1/3 & 1/3 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} x_A(k) \\ x_B(k) \\ x_C(k) \\ x_D(k) \end{pmatrix}$$

The a symptotic stationary distribution will correspond to the eigenvector of the eigenvalue $\lambda_1 = \lambda_{DOM} = 1$:

$$v_1 = v_{DOM} = \begin{pmatrix} 3 \\ 3 \\ 4 \\ 4 \end{pmatrix}$$

More explicitly, if *N* is the total number of buses, the asymptotic stationary distribution is

$$\frac{N}{14} \begin{pmatrix} 3 \\ 3 \\ 4 \\ 4 \end{pmatrix}$$

Additionally we can check that for the other eigenvalues: $|\lambda| < 1$ and each eigenvector has negative and positive coordinates.

$$\lambda_2 = 1/3 \qquad v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda_3 = -1/2 \qquad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\lambda_4 = -1/6 \qquad v_4 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

Solution of Exercise 6 (Engineering School)

(A)

(a) The equations that govern the number of students $x_i(k)$, $1 \le i \le 4$, at the course *i* and the year *k* are:

$$x_1(k+1) = 0\acute{}2x_1(k) + 600$$
$$x_2(k+1) = 0\acute{}3x_1(k) + 0\acute{}7x_2(k)$$
$$x_3(k+1) = 0\acute{}3x_2(k) + 0\acute{}7x_3(k)$$
$$x_4(k+1) = 0\acute{}3x_3(k) + 0\acute{}7x_4(k)$$

that is,

$$x(k+1) = Ax(k) + b$$

where

$$x(k) = \begin{pmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{pmatrix}, A = \begin{pmatrix} 0\acute{}2 & 0 & 0 & 0 \\ 0\acute{}3 & 0\acute{}7 & 0 & 0 \\ 0 & 0\acute{}3 & 0\acute{}7 & 0 \\ 0 & 0 & 0\acute{}3 & 0\acute{}7 \end{pmatrix}, b = \begin{pmatrix} 600 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

(b) As 1 is not an eigenvalue of *A*, there exists an unique equilibrium point $x_e = (I - A)^{-1}b$. In this case it is easy to compute it directly:

$$(x_e)_1 = 0\acute{}2(x_e)_1 + 600$$
$$(x_e)_2 = 0\acute{}3(x_e)_1 + 0\acute{}7(x_e)_2$$
$$(x_e)_3 = 0\acute{}3(x_e)_2 + 0\acute{}7(x_e)_3$$
$$(x_e)_4 = 0\acute{}3(x_e)_3 + 0\acute{}7(x_e)_4$$

which has as unique solution

$$(x_e)_i = 750, \ i = 1,...,4$$

So, in the stationary state there will be 750 students in every course, with a total number of students of 3000.

As the eigenvalues of *A* are 0´7, triple eigenvalue, and 0´2, all of them are smaller than 1, the equilibrium point is asymptotically stable, so that the solutions converge to it for any initial conditions. For example:

$$x(0) = 0 \Rightarrow x_1(1) = 600, x_1(2) = 720, x_1(3) = 744, x_1(4)749,...$$

(B)

(a) The new system is:

$$x(k+1) = A\acute{}x(k) + b$$

where

$$x(k) = \begin{pmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{pmatrix}, A\acute{} = \begin{pmatrix} 0\acute{}2 & 0 & 0 & 0 \\ 0\acute{}7 & 0\acute{}3 & 0 & 0 \\ 0 & 0\acute{}7 & 0\acute{}3 & 0 \\ 0 & 0 & 0\acute{}7 & 0\acute{}3 \end{pmatrix}, b = \begin{pmatrix} 600 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

(b) The new equilibrium point is the same than above, and it is also asymptotically stable. But now the yearly number of degree students is 525, instead of 225 above.

Solution of Exercise 7 (Fibonacci Numbers)

(a) It corresponds to a homogeneous equation in linear finite difference**s**. Its characteristic polynomial is:

$$Q(\tau) = \tau^2 - \tau - 1$$

so that the characteristic values are:

$$\lambda_1 = \frac{1+\sqrt{5}}{2}, \lambda_2 = \frac{1-\sqrt{5}}{2}$$

and the general solution is:

$$y(k) = c_1\left(\frac{1+\sqrt{5}}{2}\right)^k + c_2\left(\frac{1-\sqrt{5}}{2}\right)^k$$

where

$$0 = c_1 + c_2$$

$$1 = c_1\frac{1+\sqrt{5}}{2} + c_2\frac{1-\sqrt{5}}{2}$$

so that

$$c_1 = \frac{1}{\sqrt{5}}, c_2 = \frac{-1}{\sqrt{5}}$$

Hence,

$$y(k) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right), k = 1,2,...$$

Surprisingly, this expression gives natural number for all $k \in \mathbb{N}$: 1,1,2,3,5,…

Alternatively, it can be seen as the homogeneous discrete linear system

$$x(k+1) = Ax(k)$$

where:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

The eigenvalues of A are, equivalently:

$$\lambda_1 = \frac{1+\sqrt{5}}{2}, \lambda_2 = \frac{1-\sqrt{5}}{2}$$

so that it diagonalizes:

$$\overline{A} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = S^{-1}AS,$$

where

$$S = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}$$

and it is verified that:

$$A^k = S\begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix} S^{-1}$$

Actually, we are interested in the value of $x_1(k)$ for the initial conditions $x_1(0)=1, x_2(1)=1$, so that:

$$\begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} = S\begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix} S^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

which gives:

$$x_1(k) = \frac{1}{\sqrt{5}}\left(\lambda_1^k - \lambda_2^k\right)$$

So it coincides with the solution obtained before.

(b) It results

$$\lim_k \frac{y(k+1)}{y(k)} = \frac{1+\sqrt{5}}{2},$$

which is the famous "golden ratio". It is basic in the Greek aesthetics, and very frequent in natural phenomena as spirals in snails, plant branching, etc.

Conclusions

This paper confirms the possibility of illustrating in Engineering degrees concepts and basic results of Linear Algebra, such as: the computation of the matrix by means of consecutive states; and the eigenvectors and eigenvalues as stationary/asymptotic distribution and growing; equilibrium points and its stabity, stochastic matrices, etc. In particular, we present examples of dynamical discrete systems, which are motivating application exercises of technological disciplines. The exercises are accessible to early-year students since they are self-contained in terms of technological requirements and only basic knowledge of Linear Algebra is necessary. Furthermore, they can be implemented by means of PBL methodology.

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