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ABSTRACT

Teaching and learning in mathematics as well as research and construction of mathematical knowledge, has been the object of interest not only by educators and mathematicians but also, and especially, for philosophers. The present work is an investigation of the geometric notion of 'Riemann curvature tensor' analyzed from the theoretical-philosophical perspective of the Paradigm of Complexity, which offers us relevant epistemological principles that support our hypothesis that this notion should be conceived as a complex system and that its learning is achieved when you understand how it is generated and how it is built. This implies, in turn, that the teaching-learning processes must aim to learn to think complexly contextualizing knowledge. We will place emphasis on the transformative practical dimension of teaching that arises as a result of reflections on the teaching task itself in teacher training. We will begin with the presentation of some of the central notions of Complexity that will help us think about mathematical problems from this approach, a task that we will combine with an in-depth study of the Riemann curvature tensor; finally, we will culminate with a proposal whose objective is to adapt the ideas addressed to the field of teaching and learning geometry.

Keywords: geometric knowledge, complex system, curvature tensor

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Introduction

The Paradigm of Complexity or Complex Thought (CT) can be characterized as a theoretical-philosophical-epistemological approach committed to a vision of the world supported by the principles or macro-ideas that tries to overcome fragmentation between disciplines, generated by mechanistic conceptions and Positivists present in the tradition of Western thought. Its main representatives - which we will address in this text - are the French philosopher Edgar Morin, the Nobel Prize for Belgian nationalized Russian chemistry Ilya Prigogine and the Argentine epistemologist and meteorologist Rolando García (among other leading scientists and philosophers).

From the etymological point of view the word complexity is of Latin origin, it comes from complectere whose plectere root means 'braid, link'; and so, complexus is 'what is woven together'. This implies conceiving the phenomena to be studied as complex entities, that is, formed by multiple and heterogeneous aspects - the threads of the tissue - related to each other. For example, if we analyze the human being in any of his behaviors, the hybridity of the aspects will be that he is a being constituted by physical, social, cultural, biological nature, etc., that is, its plot includes all the dimensions that defines it.

These phenomena are technically called 'complex systems' (CS) and are considered as a complete organization of heterogeneous elements (natural, social, chemical, cultural, etc.) linked to each other by some relationship that defines the problem or object of study. In this dynamic process structure, its elements interact with each other and also with the elements that surround the problem-system, they are constituted as "open" systems, that is, permeable to changes in their environment. This emphasis placed on the link - and not on the elements - leads us to conceive that the rupture of this link is interpreted as mutilation, producing blindness in knowledge, as the main cause of the misunderstanding of phenomena. As García affirms "A CS is a representation of a cut of that reality, conceptualized as a complete organization, in which the elements are not 'separable' and, therefore, cannot be studied in isolation" (2006, p. 21)

As we said, the CT includes several logical principles that are also principles of knowledge, of which, in the present analysis, we will be limited to two general and fundamental ideas to later incorporate other concepts implied by them since they will be necessary in our approach of the tensor of curvature.

The first of these ideas can be synthesized as "the unity of the natural and the cultural." According to this approach there is a need for an articulation between knowledge and concepts that historically were separated and that now it is urgent to put them in dialogue, such as: science and philosophy, nature and spirit, reason and myth, necessity and chance, order and disorder, theory and practice, etc. Both in physical nature and in human beings, these aspects are linked in a fruitful bond. There is unity in diversity.

The resistance to these dichotomous separations is explained by another fundamental idea where it is clearly appreciated that the antagonistic is not exclusive but complementary, where it is also appreciated that opposites are needed in that incessant game typical of both natural and social or cultural processes. That second inclusive idea is the notion of ‘self-organization’. It is the recursive process that is represented very well with the image of a whirlwind which is formed with the contest of opposite flows and where each moment of the whirlwind is product and producer, is effect and cause (as for example the process of human reproduction) in a continuous process becoming.

Open complex systems are constantly exchanging energy, matter or information with the external environment; This exchange impacts its internal structure, which is sensitive to the environment, but does not lose its autonomy. While there is internal variability, as a whole it maintains a certain order, that is, that fluctuation and stability coexist. An example of such a system is the human body that as a structured totality is formed by cells, tissues, organs and systems, which interact with each other and at the same time absorb and dissipate the energy and matter of the context, which keep it alive to perform its functions. Both the external environment and the internal fluctuation behave with each other as antagonistic forces tending to generate ruptures so that if any disturbance extends the threshold of fluctuation then the system is unstructured and a new internal organization of its elements is generated with new relationships between they. These two processes of destructuring and restructuring - both synthesized in the notion of self-organization - show the non-linear evolution of certain structures. Rolando García points out in this regard that

> The evolution of such [complex] systems is not carried out through processes that are modified gradually and continuously, but proceeds by a succession of imbalances and reorganizations. Each restructuring leads to a period of relative dynamic equilibrium during which the system maintains its previous structures with fluctuations within certain limits (2000, p. 77).

These restructuring are the result of transformations where chance and necessity are the two elements that cooperatively intervene - not exclusively - and which, in turn, make it possible to introduce the notion of probability. It was Prigogine who demonstrated that in natural systems "instability can only be incorporated at a statistical level" (1995); therefore, the evolution towards a new structure in complex systems does not occur either deterministically or randomly since there are always various possibilities as candidate structures to adapt to their environment, their environment, their context. There may be many structures adaptable to the environment, and this shows that flexibility is associated with novelty and the creation of new configurations. Given the possibilities present in the fluctuations, in the bifurcations, there is also the possibility of choice, the emergence of the novelty, of a new structure; novelty is possible in conditions of instability and conflict of forces. And this movement of self-organization is present in natural phenomena - like the classic phenomenon of thermal convection of ‘instability of Bénard’ - and in social and cultural ones - like the process of knowledge construction.

We are now able to think about our object of study, the Riemann curvature tensor, as a complex system.

The Complex Construction of the Curvature Tensor

The idea of curvature contains in itself the multiplicity and diversity that knowledge presents at its different levels, and the tensor is a clear example of a complex system that has been constructed with successive structuring at its levels since its most remote beginnings in Euclid's time until today where this system acquires its most abstract formulation. In this sense, the notions of curvature and metrics are intertwined to produce a conception that transcends the precursor ideas but which in turn are contained in this emergent (the tensor).

In the historical evolution, the concept of curvature is presented explicitly with the theory of curves and surfaces, whose development is largely due to Monge and Gauss. It is Riemann who defines in an abstract way curvature tensor based on Gauss's geometric work. The curvature is already tacitly present in Euclid's fifth postulate. This postulate was a cornerstone for the further development of geometry, so towards the end of the eighteenth century, it was believed that the fifth postulate could be deduced from the previous four, perhaps adding some additional condition. The search for such a demonstration, at the beginning of the nineteenth century, generated the appearance of works such as those of Lobachevsky and Bolyai who, independently, develop the hyperbolic geometry. Gauss argued that other geometries could exist satisfying the first four postulates, but not the fifth, although he published nothing about it.

The concept of curvature, as we shall see, projects light on the question of the existence of non-Euclidean geometries. In the second half of the nineteenth century, the development of multilinear algebra made it possible to understand and formalize the curvature tensor. The curvature is present in the Riemann varieties, in the theory of relativity and in geometric structures such as symmetric and homogeneous spaces. To obtain a reasonable modeling of the world in which we live, it is not enough with linear models, it is necessary to introduce objects formed with higher order terms. The concept of curvature is precisely a second order entity, which arises naturally in the study of curves, surfaces and their generalizations.

Curvature also plays a fundamental role, both in physics and in other experimental sciences. For example, the magnitude of the force required to move an object at constant speed is, according to Newton's laws, a constant multiple of the curvature of the trajectory; or the movement of a body in a gravitational field is determined, according to Einstein, by the curvature of spacetime. Morin's great contribution is to have managed to synthesize various trends in current science at a higher level of integration while respecting the specificity and achievements of each of them. In this sense we think that the concept of curvature tensor given by Riemann in his research plan "On the Hypotheses which lie at the Bases of Geometry" of 1854, meets this expectation.

The CT explains this interdisciplinary integration in terms of the interactions that the CS - the tensor - has with its environment, overcoming the hyperspecialization that leads to the fragmentation and division of knowledge in watertight compartments, and thereby achieving a goal point of view that promotes communication, dialogue, the round trip of the productive circle between inside and outside the frontiers of science. We will make an analysis in the sense of demonstrating the hypothesis stated above, about Riemann's research plan. According to Morin, a reform of thought is necessary whose task is not to accumulate knowledge in terms of systems and totality, as has been done, but in terms of organization and articulation, which leads not so much to fix the totality of knowledge in each discipline, but in crucial knowledge, strategic points, communication nodes, organizational articulations between disjoint orbits (1993, p. 19).

Taking into account this premise, which reaffirms the dialogic communication and the fruitful and solidary exchange of concepts, we think that Riemann's research plan is an enlightening example in this regard because, as we will see there is, on the one hand, organizational articulation between disjointed orbits when in The Application to Space section of his research plan anticipates the bases of the theory of general relativity. In addition, he detected that discrete quantities would be required for the domain of small distances, that is, the need for quantum mechanics. This last observation that derives from Riemannian geometry puts us in the presence of crucial knowledge, a strategic inflection point, and a knot of communication between different disciplines such as mathematics and theoretical physics.

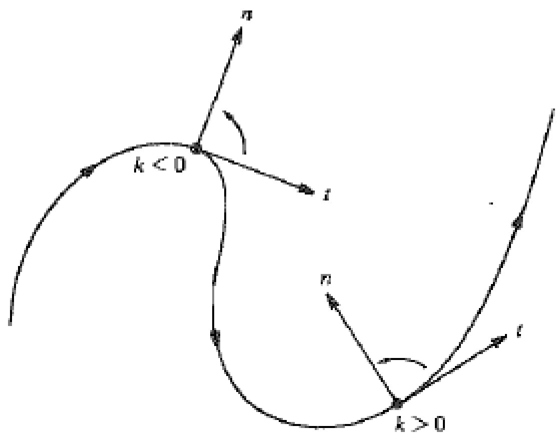
The Concepts of Metric and Curvature

This section is a preamble to the analysis of Riemann's work "On the Hypotheses which lie at the Bases of Geometry". It is intended to provide the reader not familiar with the concepts of differential geometry of the elements that allow understanding the analysis of the work cited. The concepts of metric, curvature and curvature tensor of Riemann are approached intuitively and we minimizing technicalities.

The Curvature in Curves and Surfaces

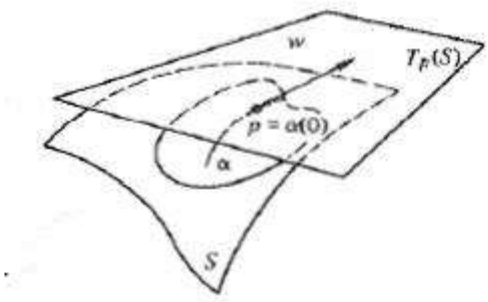
If the movement of a particle (material point) in the plane or in the three-dimensional space is considered and the position of said point is observed for each moment of time, a plane or space curve is obtained. The curves are one-dimensional entities (manifols). The velocity of this point is the instantaneous variation rate with respect to the time of the position and it is described by a vector that is tangent to the curve. The curvature (which it is denoted by κ) at a given point measures the deviation experienced by the curve with respect to its tangent vector. This curvature is measured by a function that depends on the point namely the norm of acceleration (second order entity), can be positive or zero. The curvature is a geometric invariant of the curve, that is, it will be the same for corresponding points of congruent curves. Congruent curves are obtained by performing rigid movements (isometries) in the plane or space.

Figure 1. *Curvature for a Curve*[1]



A surface is a two-dimensional object such as a sheet of metal (of course, this has a thickness, for our study we will assume that this thickness is null). If we consider the unfolded sheet, we are in the presence of a portion of the Euclidean plane, but when we submit it to non-reversible deformations appear a distortion that is measured precisely by the curvature. Now the curvature of the given point surface is the product of two numbers called principal curvatures, this product is called Gaussian curvature and is designated by K. From an intuitive point of view, it can be said that the curvature of a surface at one point it measures its deviation from the plane tangent to the surface at that point.

Figure 2. *Curvature on a Surface*



The theory of surfaces of the Euclidean space has developed fundamentally throughout the eighteenth, nineteenth and first half of the twentieth centuries. It is noteworthy the contributions due to Euler, Monge and Dupin. Johann Carl Friedrich Gauss's article *Disquisitiones generales circa superficies curvas* appeared in 1827, was fundamental to lay the concept of space on a solid basis. It also introduces the notion of curvature of a surface at a point, which is an intrinsic concept. In this treatise the famous Gaussian Theorem (Egregium Theorem)

[1]The graphics included in this work are taken from Do Carmo M. (2010). Differential Geometry of Curves and Surfaces.

appears that states: At one point on the surface, the curvature of Gauss is an isometric invariant. Informally, the theorem says that the Gaussian curvature of a surface can be determined entirely by measuring angles and distances on the surface itself, without referring to the particular way in which it curves within the three-dimensional Euclidean space. The theorem can be used to see that two surfaces are not isometric. For example there can be no isometry between the plane and the sphere, not even a portion of it. It is known that the Gaussian curvature of the sphere of radius r is, $K_{sphere} = 1/_{r^2} > 0$ while for the plane it is $K_{plane} = 0$, if an isometry exists, the Gaussian curvature should be preserved. This says that any flat representation of the earth is necessarily distorted.

Riemann Geometry

In 1854 George Friedrich Bernhard Riemann generalizes Gauss's ideas to spaces of a dimension greater than three in the famous report "On the Hypotheses which lie at the Bases of Geometry" published posthumously. In this dissertation Riemann presents the concept of differentiable manifold as an n-dimensional set on which the calculations of the ordinary analysis can be performed; In this context, curves and surfaces are one-dimensional and two-dimensional varieties respectively. For Riemann, to give a geometry over a variety is to define a positive definite quadratic form in each of the tangent spaces. This definition of Riemann makes it possible to extend Gauss's work to a good extent. Being a generalization, the Riemann spaces of variable curvature comprise as particular cases the constant curvature spaces, which are those that historically gave rise to non-Euclidean geometries and Euclidean geometry. The properties of the curvature tensor are quite complicated; however, his main ideas are original and subtle. Riemann presents a broad generalization of all known geometries, both Euclidean and non-Euclidean, in natural language and without technicalities. This field is known today as Rimannian Geometry, and apart from its importance in pure mathematics, it proved to be the appropriate mathematical scaffolding for Einstein's theory of general relativity. The generalization given by Riemann, highlights two nuclear concepts that are the metric and the curvature of a manifold.

<u>Metric</u>

Let's see how the concept of metric given by Gauss generalizes. It is known that a surface that is in three-dimensional space can be expressed parametrically by three functions that depend on certain parameters u and v. A point on the surface is determined by three functions (called coordinate functions) $x = x(u, v)$, $y = y(u, v)$ and $z = z(u, v)$ the u and v parameters can be interpreted as coordinates of the surface points. The distance ds between two near points (u, v) and $(u + du, v + dv)$ along the surface, is given by a quadratic differential form, namely:

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

where E, F, and G are certain functions of u and v. This differential form allows calculating the length of curves on surfaces, finding the geodesic curves (the shortest ones) and calculating the Gaussian curvature of the surface at any point, all without reference to the ambient space. Riemann, generalized this by discarding the notion of ambient space and introducing the notion of continuous n - dimensional variety of points $(x_1, x_2 \cdots, x_n)$. A distance or metric ds between close points $(x_1, x_2 \cdots, x_n)$ and $(x_1 + dx_1, x_2 + dx_2 \cdots, x_n + dx_n)$ is a quadratic differential formula:

$$ds^2 = \sum_{i,j} g_{ij} dx_i dx_j \quad (1)$$

where the g_{ij} are appropriate functions of $x_1, x_2 \cdots, x_n$. Different systems of g_{ij} define different Riemannian geometries on the manifold. The metric thus defined makes it possible to measure the length between two points of a curve that rests on the manifold, the angle between curves and other geometric entities. In other words, the metric is what allows us to make geometry.

Curvature

Another concept analyzed by Riemann is the curvature for these manifolds and he investigated the special case of constant curvature. He introduced the concept of curvature tensor, which is reduced to Gaussian curvature for $n = 2$ and whose annulment proved necessary and sufficient for the given quadric metric to be equivalent to the Euclidean. From this point of view, the curvature tensor measures the deviation of the Riemannian geometry defined by the formula (1) with respect to the Euclidean geometry. The concept of Gaussian curvature of a surface extends to Riemann manifolds of a dimension greater than two in a natural way, since it is possible to consider the germ of the totally geodetic surface tangent at a point of manifold to the subspace of dimension two. The Gaussian curvature of said surface is defined as the sectional curvature of the plane at said point. In general, the curvature tensor of a Riemannian manifold depends on four arguments, while the sectional curvature only two. This result may seem strange, although it is now well known that in a Riemannian manifold the knowledge of sectional curvature at one point determines that of the curvature tensor.

Non-Euclidean Geometries

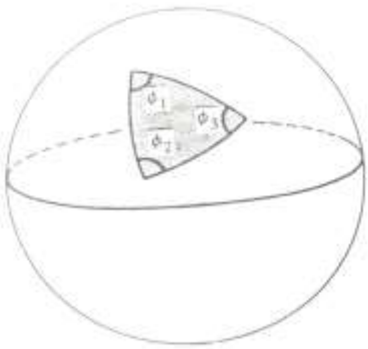
The historical development of the curvature was influenced by the fifth postulate of the Elements of Euclid whose statement is:

> "In a plane, given a line and a point not on it, at most one line parallel to the given line can be drawn through the point."

Line segments in Euclidean geometry have the property of making the minimum distance between two given points.

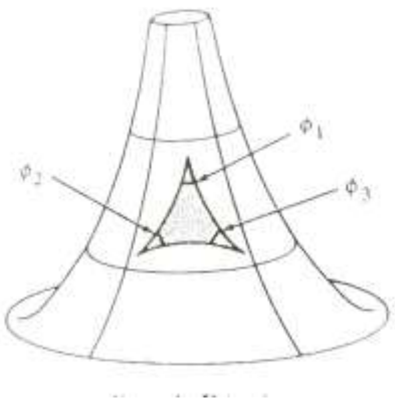
If the case of a sphere is considered, it is known that the shortest distance between two points on the sphere is the meridian arcs, which can be considered straight in the sphere. When considering a line on the sphere and a point outside it, there is no other that passes through that point and does not cut to the first. That is to say that in this case, the fifth postulate is not verified. In the spheres the sum of the internal angles of a triangle is greater than two right angles (unlike a flat triangle whose sum of interior angles is two right), the excess is due to its curvature.

Figure 3. *Angles in the Sphere*



It is possible to define Hyperbolic Geometry as one that satisfies all trigonometric formulas of a spherical geometry in which the radius is pure imaginary. Minding discovered the pseudosphere (surface of revolution of the tractrix) that, locally, has the properties of the hyperbolic plane. In the case of the pseudosphere the sum of the internal angles of a triangle is less than two right angles. Riemann proves how the sphere can be assigned a quadratic form with coefficients that are functions of the coordinates and with positive curvature. He explicitly states that the geometry that everyone had sought, hyperbolic geometry, is defined by the same quadratic shape with negative curvature. The interpretation could hardly be simpler. In the case of Euclidean geometry, as already said, the curvature is zero.

Figure 4. *Angles in the Pseudosphere*



On the Hypotheses which Lie at the Bases of Geometry

Uber die Hypothesen, welche der Geometrie zu Grunde liegen (On the Hypotheses which lie at the Bases of Geometry) is the title of the habilitation dissertation presented to the Faculty of Philosophy of the University of Göttingen published after the death of its author in 1854. In this conference (Riemann) despite not having a detailed definition of n-dimensional manifold (in the sense that we know it today), he introduced the concept of Riemannian metric, explains the curvature tensor, sectional curvature, and offers some relations between the metric and the curvature. In his presentation, he gives his research plan, which consists of the following sections:

I) Notion of an *n-ply* extended magnitude.
II) Measure-relations of which a manifoldness of n dimensions is capable on the assumption that lines have a length independent of position, and consequently that every line may be measured by every other.
III) Application to Space.

In the section *Notion of an n-ply extended magnitude* of his work, Riemann introduces the concept he calls extended n-dimensional manifold. According to the above, an *n*-dimensional manifold is a set in which, every point is completely determined by *n* numbers, this notion is local, that is, it is valid in the neighborhoodof the point in question. The following paragraphs are from the section cited and in which the definition of the concept of variety is addressed.

> If in the case of a notion whose specialisations form a continuous manifoldness, one passes from a certain specialisation in a definite way to another, the specialisations passed over form a simply extended manifoldness, whose true character is that in it a continuous progress from a point is possible only on two sides, forwards or backwards. If one now supposes that this manifoldness in its turn passes over into another entirely different, and again in a definite way, namely so that each point passes over into a definite point of the other, then all the specialisations so obtained form a doubly extended manifoldness. In a similar manner one obtains a triply extended manifoldness, if one imagines a doubly extended one passing over in a definite way to another entirely different; and it is easy to see how this construction may be continued. If one regards the variable object instead of the determinable notion of it, this construction may be described as a composition of a variability of $n + 1$ dimensions out of a variability of n dimensions and a variability of one dimension.
> In other words, let us take a continuous function of position within the given manifoldness, which, moreover, is not constant throughout any part of that manifoldness. Every system of points where the function has a constant value, forms then a continuous manifoldness of fewer dimensions than the given one. These manifoldnesses pass over continuously into one another as the function changes; we may therefore assume that out of one of them the others proceed, and speaking generally this may occur in such a way that each point passes over into a definite point of the other; the cases of exception (the study of which is important) may here be left unconsidered. Hereby the determination of position in the given manifoldness

> is reduced to a determination of quantity and to a determination of position in a manifoldness of less dimensions. It is now easy to show that this manifoldness has *n* - 1 dimensions when the given manifold is *n*-ply extended. By repeating then this operation *n* times, the determination of position in an *n*-ply extended manifoldness is reduced to *n* determinations of quantity, and therefore the determination of position in a given manifoldness is reduced to a finite number of determinations of quantity *when this is possible*.[2]

In these paragraphs, Riemann's concern for defining the concept of manifold is perceived, this is motivated by the fact that he wanted to define an appropriate model for the universe. In Riemann's conception the universe is not contained within another ambient space. It is noted that with the notion of extended n-manifold the concepts of lines and surfaces are generalized, these entities under the new look are 1 and 2 - dimensional manifolds respectively. Let us observe that the effort to conceive an idea of space that allows us to have a more delimited conception of our universe is the pulsor that puts the holgrammatic operator in synergy because, in the generalization that Riemann calls extended n- manifold, it is clear that not only the parts are within the whole, but the whole is within the parts. Each extended n- manifold can be considered as a whole that contains smaller n- manifolds. The hologrammatic principle, which explains the relationships between varieties, is one of the most fundamental ideas of the CT because it is closely linked to the idea of organization. The hologram being a physical image of an object "each point of the hologram object is memorized throughout the hologram, and each point of the hologram contains the presence of the object in its entirety, or almost" (Morin, 2010, p. 112). An interesting aspect that shows this principle is the ability of the parties to regenerate the whole, which is more clearly visualized in the organization of living beings whose cells, being controlled by the whole organism, in turn contain the information of the whole be the one they are able to produce again.

The Definition of Metric

Having defined the concept of extended n- manifold, Riemann considers the problem of establishing a metric on the manifold. In section two of his work he presents the definition of metric. Although it did not have the notion of space tangent to the manifold as we know it today; Riemann makes the basic and crucial observation that to find the length of a curve on a variety, it is enough to know how to calculate the norm of the velocity vectors at each point of the curve. Therefore, if a norm is defined (and therefore an internal product), in each tangent space, the length of any curve can be calculated. Riemann assumes that this norm varies continuously with respect to points in the manifold, and also assumes that this norm comes from an internal product as the following paragraphs in section two:

[2]All citations in this section are the English traslation that appear in: W. K. Clifford, "On the Hypotheses lie at the Bases of Geometry. Nature. 8 (183–184), 14–17, 36, 37, unless otherwise specified.

> The hypothesis which first presents itself, and which I shall here develop, is that according to which the length of lines is independent of their position, and consequently every line is measurable by means of every other. Position-fixing being reduced to quantity-fixings, and the position of a point in the *n*-dimensioned manifoldness being consequently expressed by means of *n* variables $x_1, x_2, x_3, \ldots, x_n$, the determination of a line comes to the giving of these quantities as functions of one variable. The problem consists then in establishing a mathematical expression for the length of a line, and to this end we must consider the quantities x as expressible in terms of certain units. I shall treat this problem only under certain restrictions, and I shall confine myself in the first place to lines in which the ratios of the increments dx of the respective variables vary continuously. We may then conceive these lines broken up into elements, within which the ratios of the quantities dx may be regarded as constant; and the problem is then reduced to establishing for each point a general expression for the linear element ds starting from that point, an expression which will thus contain the quantities x and the quantities dx. I shall suppose, secondly, that the length of the linear element, to the first order, is unaltered when all the points of this element undergo the same infinitesimal displacement, which implies at the same time that if all the quantities dx are increased in the same ratio, the linear element will vary also in the same ratio. On these suppositions, the linear element may be any homogeneous function of the first degree of the quantities $dx,$ which is unchanged when we change the signs of all the $dx,$ and in which the arbitrary constants are continuous functions of the quantities $x.$
>
> This differential expression, of the second order remains constant when *ds* remains constant, and increases in the duplicate ratio when the *dx*, and therefore also *ds*, increase in the same ratio; it must therefore be ds^2 multiplied by a constant, and consequently ds is the square root of an always positive integral homogeneous function of the second order of the quantities dx, in which the coefficients are continuous functions of the quantities x. For Space, when the position of points is expressed by rectilinear co-ordinates, $ds = \sqrt{dx^2}$ Space is therefore included in this simplest case.
>
> I restrict myself, therefore, to those manifoldnesses in which the line element is expressed as the square root of a quadric differential expression.
>
> Manifoldnesses in which, as in the Plane and in Space, the line-element may be reduced to the form $\sqrt{dx^2}$, are therefore only a particular case of the manifoldnesses to be here investigated; they require a special name, and therefore these manifoldnesses in which the square of the line-element may be expressed as the sum of the squares of complete differentials I will call *flat*.

In the last paragraph Riemann makes us observe that the degree of generality of the metrics with which the Riemannian manifold can be provided. The concept of curvature also appears implicitly in this paragraph because as we will see the fl at spaces are those with zero curvature. The dual interplay of metric and nodal point curvature in Riemann’s theory makes its appearance. The author realizes that in order to continue studying the properties of the metric it is necessary to limit the flexibility of the coordinate change, and take some coordinates that are constructed in a geometric way, enter the geodesic coordinates as explained in the following paragraph:

> For this purpose let us imagine that from any given point the system of shortest limes going out from it is constructed; the position of an arbitrary point may then be determined by the initial direction of the geodesic in which it lies, and by its distance measured along that line from the origin.

The Notion of Curvature

In the following paragraphs in section two, Riemann explains the notion of curvature using Gauss's Theorem Egregium. Here he exposes the geometric meaning of the curvature of a manifold. A precise definition of the curvature occurred years later, it was necessary to introduce the notion of tensor, forged mainly by Ricci.

> In the idea of surfaces, together with the intrinsic measure-relations in which only the length of lines on the surfaces is considered, there is always mixed up the position of points lying out of the surface. We may, however, abstract from external relations if we consider such deformations as leave unaltered the length of lines - *i.e.*, if we regard the surface as bent in any way without stretching, and treat all surfaces so related to each other as equivalent. Thus, for example, any cylindrical or conical surface counts as equivalent to a plane, since it may be made out of one by mere bending, in which the intrinsic measure-relations remain, and all theorems about a plane - therefore the whole of planimetry - retain their validity. On the other hand they count as essentially different from the sphere, which cannot be changed into a plane without stretching. According to our previous investigation the intrinsic measure-relations of a twofold extent in which the line-element may be expressed as the square root of a quadric differential, which is the case with surfaces, are characterised by the total curvature. Now this quantity in the case of surfaces is capable of a visible interpretation, viz., it is the product of the two curvatures of the surface, or multiplied by the area of a small geodesic triangle, it is equal to the spherical excess of the same. The first definition assumes the proposition that the product of the two radii of curvature is unaltered by mere bending; the second, that in the same place the area of a small triangle is proportional to its spherical excess. To give an intelligible meaning to the curvature of an *n*-fold extent at a given point and in a given surface-direction through it, we must start from the fact that a geodesic proceeding from a point is entirely determined when its initial direction is given. According to this we obtain a determinate surface if we prolong all the geodesics proceeding from the given point and lying initially in the given surface-direction; this surface has at the given point a definite curvature, which is also the curvature of the *n*-fold continuum at the given point in the given surface-direction is given.

Today the curvature that Riemann defines is known as the sectional curvature, and this one assigns to each two-dimensional subspace of the tangent space a real number. It should be noted that instead of talking about the curvature in the surface direction, we talk about the sectional curvature in a two-dimensional subspace. As we had pointed out, the curvature tensor of a Riemannian manifold depends on four arguments, while the sectional curvature only two. And we had also said that, in a Riemannian manifold, knowledge of sectional curvature at one point determines the curvature tensor. This leads us to think about how the organizational recursion operator acts on the idea of Riemann just mentioned. Here it is observed how the knowledge of the parties is related to the knowledge of the whole, how new qualities arise that did not exist in the isolated parts, that is, they are the organizational emergencies that are not deduced from the previous elements. In the process of recursive organization, the phenomena - complex systems - are explained by the tetragram proposed by Morin: order/disorder/ interaction/organization, where there is no primacy of one over others but they are interdependent (2008, p. 150). As we said at the beginning, each restructuring is the evolution towards a new structure that turns out to be the realization of one of the multiple probabilities that nonlinear causality offers the system. This new organization appears with original, unpublished characteristics, because changes are generated in the relations between the elements and therefore it turns out to be a novelty, an emergency, a creation. The fact that there are several possibilities shows the flexibility of the structures in the face of fluctuations and it is these bifurcations that allow a choice.

This flexibility in the field of mathematical ideas has also taken the name of polysemy or ambiguity. Emily Grosholz proposes that polysemy does not generate confusion but creates the conditions for the generation of new ideas by stating that "when different representations are juxtaposed and superimposed, the result is often a productive ambiguity that expresses and generates new knowledge" (2007, p. 25); That new knowledge is a new structure. W. Byers has referred to this concept as a metaphorical quality characteristic of numerous mathematical situations where antagonistic ways of approaching a topic are proposed, and for this reason they become a matrix of deep ideas. For Byers “ambiguity is not only present in mathematics, it is essential. Ambiguity, which implies the existence of multiple, conflicting frames of reference, is the medium in which new mathematical ideas arise” (2007, p. 23).

Geometries and the Notion of Curvature

In general, it can be said that Euclidean geometry is defined as the totality of the concepts that are conserved by rigid movements in the Euclidean space (isometries). In the following section, Riemann explains that manifols with constant sectional curvature reflect one of the essential properties of a geometry. That is, the property of the invariance of objects under isometries.

> Manifoldnesses whose curvature is constantly zero may be treated as a special case of those whose curvature is constant. The common character of those continua whose curvature is constant may be also expressed thus, that figures may be viewed in them without stretching. For clearly figures could not be arbitrarily shifted and turned round in them if the curvature at each point were not the same in all directions. On the other hand, however, the measure-relations of the manifoldness are entirely determined by the curvature.

The non-Euclidean geometry constructed by Lobachevsky and Bolyai around 1829 (independently), held that through an outside point to a straight line passed more than one parallel. When Riemann discloses the assumptions that underlie Geometry, there was still confusion about non-Euclidean geometries and the examples of Lobachevsky and Bolyai were not fully accepted. According to Morin, confusion and uncertainty are not the last words of knowledge, but the precursor signs of complexity (1993, p. 30). As we can see in the previous quotes by Grosholz and Byers, we can argue that the aforementioned confusion amounts to a productive ambiguity. Confusion and uncertainty should not be taken here as psychological states but as essential cognitive instruments of the epistemological paradigm of complexity. On the other hand, the notion of uncertainty is associated with the notion of probability and, therefore, with random phenomena. That is to say that in any process the restructuring does not take place in a deterministic way towards a single possible state, but that there is a degree of uncertainty about what the ‘chosen’ structure will be among the various probable candidates; and this is valid for both natural and social phenomena and for knowledge - as in our case -.

Using the notion of Riemannian geometry, Riemann gives a first concrete example for a non-Euclidean geometry, as the following paragraph in section two.

> The measure-relations of these manifoldnesses depend only on the value of the curvature, and in relation to the analytic expression it may be remarked that if this value is denoted by α, the expression for the line-element may be written $\frac{1}{1+\frac{\alpha}{4}\|x\|^2}\sqrt{\sum dx_i^2}$

It follows that the example constructed by Lobachevsky and Bolyai is obtained assuming α constant and negative, i.e it results a geometry according to Riemann. In the last two paragraphs we have just quoted and commented on, the very meaning of the word complex, unity in diversity is highlighted. Riemann puts us in direct contact with complex thought by capturing the diversity and plurality of unity, that is, with a thought that links and globalizes.

Curvature and Interdisciplinary Complexity

In the new disciplines its specialization is interdisciplinary, so that the disciplinary closure requires at the same time the opening to other disciplines. This can be observed in the Applications to space section where Riemann implicitly states that the basic purpose of the ideas created was to understand the space where we live. In this section there is an opening towards theoretical physics transcending the boundaries of Riemannian geometry and anticipating the ideas of the theory of relativity and quantum mechanics. For Riemann our universe had no zero curvature, that is, it is not a flat manifold. According to Riemann, the metric of space should be searched for in physical properties, that is, of the observation, as can be seen in the following paragraph.

> In the course of our previous inquiries, we first distinguished between the relations of extension or partition and the relations of measure, and found that with the same extensive properties, different measure-relations were conceivable; we then investigated the system of simple size-fixings by which the measure-relations of space are completely determined, and of which all propositions about them are a necessary consequence; it remains to discuss the question how, in what degree, and to what extent these assumptions are borne out by experience.

It follows that, for Riemann, the space metric had to be determined by observing nature. For him, the curvature determines the metric and the metric determines the curvature. In the General Theory of Relativity, from observation it follows what is known as the energy-momentum tensor, knowing this tensor determines the curvature of space and this knowledge of curvature determines the metric, as Riemann claimed. This shows that the substantial idea of the Theory of Relativity is present in the foundations of Riemann's geometry. For physical reasons, the General Theory of Relativity considers time as part of the manifold which implies studying a manifold of dimension four, that is, space-time, rather than a three-dimensional manifold. Time as a new coordinate of the manifold does not behave like the others, since the movement in three-dimensional space is reversible, but we cannot return in time. Semi-Riemannian metrics have the distinction of distinguishing the time variable from the spatial ones. In a semi-Riemannian metric the condition of being a positive definite bilinear form is changed, by the condition of being a non-degenerated bilinear form.

Every organization gives rise to new qualities that did not exist in isolated parts, they are the organizational emergencies that are not deduced from the previous elements. This is evident in the following elucidation of Riemann, where he speculates that it is feasible that our universe is finite, in the sense that it has a finite diameter.

> The unboundedness of space possesses in this way a greater empirical certainty than any external experience. But its infinite extent by no means follows from this; on the other hand if we assume independence of bodies from position, and therefore ascribe to space constant curvature, it must necessarily be finite provided this curvature has ever so small a positive value.

It is remarkable that this universe model was suggested by Einstein many years later. Morin points out that "in the beginning was complexity" (1993, p. 77) to highlight how the very foundation of reality is not simplicity but complexity. The concept of complexity sees the systematic and multidimensional phenomena. The following quote shows that Riemann sensed that Riemannian geometry should have its reservations when modeling the physics of small distances.

> The questions about the infinitely great are for the interpretation of nature useless questions. But this is not the case with the questions about the infinitely small. It is upon the exactness with which we follow phenomena into the infinitely small that our knowledge of their causal relations essentially depends.
>
> Now it seems that the empirical notions on which the metrical determinations of space are founded, the notion of a solid body and of a ray of light, cease to be valid

> for the infinitely small. We are therefore quite at liberty to suppose that the metric relations of space in the infinitely small do not conform to the hypotheses of geometry; and we ought in fact to suppose it, if we can thereby obtain a simpler explanation of phenomena.
> The question of the validity of the hypotheses of geometry in the infinitely small is bound up with the question of the ground of the metric relations of space. In this last question, which we may still regard as belonging to the doctrine of space, is found the application of the remark made above; that in a discrete manifoldness, the ground of its metric relations is given in the notion of it, while in a continuous manifoldness, this ground must come from outside. Either therefore the reality which underlies space must form a discrete manifoldness, or we must seek the gound of its metric relations outside it, in binding forces which act upon it.

As evidenced in the cited paragraphs, Riemann envisioned that in the domain of the immeasurably small it was forced to consider discrete quantities. This refers us to the nodal idea of quantum mechanics where electromagnetic radiation is absorbed and emitted by matter in the form of quanta, that is, a discrete entity.

Final Considerations

The present work constitutes a reflection on our own pedagogical practices as teacher trainers, has motivated us and oriented to the attempt to carry out an interdisciplinary work with the objective not only of transmitting the specific disciplinary content that we have developed here, but also, and in a way intertwined, to put into play available to students the epistemological foundations that validate that content.

In this way we find ourselves in a goal that exceeds fragmentation, where we first had to establish certain ideological agreements about our conception of the world and the values committed to it, that is what Garcia calls "the epistemic framework of the interdisciplinary research" (2006, p. 35).

Due to the timeliness of the objective, the interdisciplinary work was made up of the areas of Philosophy, Geometry and Education, and we soon noticed the relevance of this integration since the concept of Riemman's curvature tensor is (as we argued) a complex system whose main thread-elements correspond to those disciplines, and being a notion of a high level of abstraction, their understanding requires an additional effort to contextualize it significantly.

Following the educational guidelines of Morin, who in his text on education *The well-placed head* proposes us the challenge of teaching to state and solve problems based on organizational principles of knowledge - such as self-organization, hologram, recursive and dialogic -, and not as an accumulation of data or information, we set out to incorporate these principles by articulating them with the subject of study and opening the thought to the context showing that it is constitutive of each phenomenon, which cannot be conceived in isolation in a pure abstraction.

Our task was guided by the premise that "the development of the ability to contextualize and totalize knowledge becomes an imperative of education"

(Morin, 1999, p. 27), and this is complex thought, which implies unite the scientific culture with the humanities.

Finally, the great challenge that Complexity gives us to trainers of trainers, professors of future professors, is the search for strategies that guide the dialogue between the knowledge that appears scattered; there is no list of rules or strategies set in advance to follow and according to Morin, everyone must find and create in their own field those paths; the present work is our proposal, a possible way.

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