On Ratios of Areas Related to the Routh-Steiner Theorem: Exploring Activities

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Abstract

Coxeter considered the following theorem of affinegeometry: If the sides BC, CA, AB of a triangle ABC are divided at L, M, N in the respective ratios $\lambda : 1, \mu : 1, \nu : 1$, thecevians AL, BM, CN form a triangle whose area is

$$\frac{(\lambda\mu\nu-1)^2}{(\lambda\mu+\lambda+1)(\mu\nu+\mu+1)(\nu\lambda+\nu+1)}$$

times that of ABC.

Coxeter attributed this theorem to Routh and mentioned also that Steiner had rediscovered it. He gave a general proof of this result using barycentriccoordinates. The search after nice expressions of area ratios succeeds in symmetric divisions or in special cases. In particular, if each side of the triangle is divided into n equal parts then nice expressions are obtained for area ratios of a hexagon and the triangle. The theorem of Marion Walter is a special case: If the trisection points of the sides of any triangle are connected to the opposite vertices, the resulting hexagon has one-tenth the area of the original triangle. The theorem of Morgan is a generalization: If we divide the side of a triangle into n equal parts, where n is odd, then the ratio of the area of the hexagon, formed by connecting the two cevians from each vertex to the central points on the opposite side, to the area of the triangle is $\frac{8}{9n^2-1}$.

These cases and others will be core of some exploring activities based directly on the Routh-Steiner theorem or its generalization.

Keywords: Ratios of areas; Routh-Steiner theorem; Barycentric coordinates; Golden ratio; Patterns

Introduction

Coxeter(1989, p. 211) in his book, considered the following theorem of *affine* geometry;

Theorem1 *If the sides* BC, CA, AB *of a triangle* ABC *are divided at* L, M, N *in the respective ratios* $\lambda : 1, \mu : 1, \nu : 1,$ *the cevians* AL, BM, CN *form a triangle whose area is*

$$\frac{(\lambda\mu\nu-1)^2}{(\lambda\mu+\lambda+1)(\mu\nu+\mu+1)(\nu\lambda+\nu+1)}$$

times that of ABC.

He emphasized that this result was discovered by Steiner, but simultaneously cited two references the first was Steiner's work (Steiner,1882) and the second was Routh's work (Routh, 1896). Later in his book he referred to the result as "Routh's theorem" (Coxeter, 1989, p. 219) admitting to the contribution of both scientists in revealing the theorem.

Coxeter gave a general proof of this result using *barycentric* coordinates attributed to Möbius. These are *homogeneouscoordinates* (t_1, t_2, t_3), where t_1, t_2, t_3 are masses at the vertices of a triangle of reference $A_1A_2A_3$. In particular $(1,0,0)$ is A_1, $(0,1,0)$ is A_2, $(0,0,1)$ is A_3 and (t_1, t_2, t_3) corresponds to a point P such that the areas of the triangles $PA_2A_3, PA_3A_1, PA_1A_2$ are proportional to the barycentric coordinates t_1, t_2, t_3 of P, respectively (see Fig. 1). If $t_1 + t_2 + t_3 = 1$ then the normalized barycentric coordinates (t_1, t_2, t_3) are called *areal* coordinates. In this case the areas of the triangles $PA_2A_3, PA_3A_1, PA_1A_2$ are t_1, t_2, t_3 times the area of the whole triangle $A_1A_2A_3$, respectively.

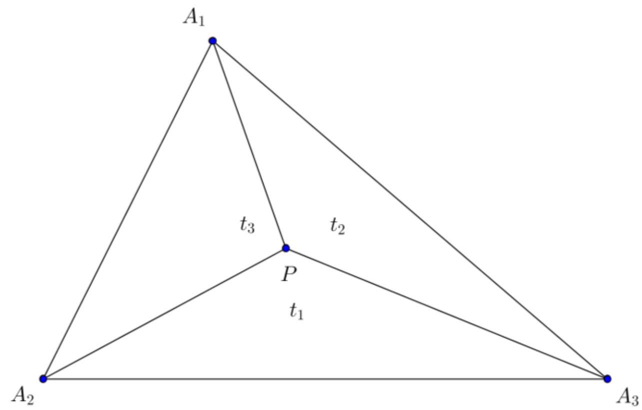

Figure 1: The areas of the triangles are proportional to the barycentric coordinates

In this paper will shall focus on some enriching activities dealing with ratios of areas based directly on the Routh-Steiner Theorem or its generalization(Abboud, 2015).

Golden ratios

We shall explore ratios of areas when we divide the sides of the triangle by Fibonacci numbers. Recall that the Fibonacci sequence is defined by the following recurrence relation:

$$f_n = f_{n-1} + f_{n-2}$$
$$f_0 = 0,\ f_1 = 1.$$

Thus, the Fibonacci numbers for $n = 0,1,2,...$ are

$$0,1,1,2,3,5,8,13,21,34,55,89,144,233,...$$

The ratios of successive Fibonacci numbers $\frac{f_n}{f_{n-1}}$ approaches the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$ as n approaches infinity, as first proved by Scottish mathematician Robert Simson in 1753 (Wells, 1986). The golden ratio satisfies the following relation

$$\varphi^2 = \varphi + 1.$$

By induction we may prove for $k \geq 2$,

$$\varphi^k = \varphi^{k-1} + \varphi^{k-2} = \varphi f_k + f_{k-1}. \qquad (1)$$

Suppose now that, for fixed $k \geq 1$, each side of a given triangle ABC is divided by the ratio $f_{n+k} \ : \ f_n$. Clearly, we have

$$\lim_{n\to\infty} \frac{f_{n+k}}{f_n} = \varphi^k.$$

On the other hand, by Theorem 1, the area ratio of the formed triangle Δ to the triangle ABC is given by the following relation:

$$\frac{S_\Delta}{S_{ABC}} = \frac{((\frac{f_{n+k}}{f_n})^3 - 1)^2}{((\frac{f_{n+k}}{f_n})^2 + \frac{f_{n+k}}{f_n} + 1)^3} = \frac{(\frac{f_{n+k}}{f_n} - 1)^2}{(\frac{f_{n+k}}{f_n})^2 + \frac{f_{n+k}}{f_n} + 1}.$$

In particular,

$$\lim_{n\to\infty} \frac{S_\Delta}{S_{ABC}} = \frac{(\varphi^k - 1)^2}{\varphi^{2k} + \varphi^k + 1} = 1 - 3\frac{\varphi^k}{\varphi^{2k} + \varphi^k + 1}.$$

Equivalently, by Eq. 1, we have:

$$\lim_{n\to\infty} \frac{S_\Delta}{S_{ABC}} = 1 - 3\frac{f_k\varphi + f_{k-1}}{(f_{2k} + f_k)\varphi + f_{2k-1} + f_{k-1} + 1}. \qquad (2)$$

Table 1,shows some values of this limit.

Table 1:Limit of area ratios

k	$\lim_{n\to\infty} \frac{S_\Delta}{S_{ABC}}$	$\lim_{n\to\infty} \frac{S_{ABC}}{S_\Delta}$
1	0.07	13.71
2	0.25	4
3	0.45	2.21
4	0.625	1.6
5	0.75	1.33
6	0.84	1.19
7	0.9	1.11
8	0.94	1.07

We can construct examples and verify this table using any dynamic geometry software. Ifthe sides of the triangle ABC are divided by the ratios

$f_5 : f_1 = 5 : 1$ or $f_{12} : f_8 = 144 : 21$, then $\frac{S_{ABC}}{S_\Delta}$ will be 1.9375 or 1.59964, respectively. It is remarkable that for $k = 4$, the sequence of divisions $f_5 : f_1,\ f_6 : f_2,\ f_7 : f_3,\ ...,\ f_{n+4} : f_n$ implies that $\frac{S_{ABC}}{S_\Delta}$ approaches 1.6 rapidly as n approaches infinity. In addition, among all the values of k in Table 1, $k = 4$ gives the best possible approximation of $\frac{S_{ABC}}{S_\Delta}$ to the golden ratio.

Another important observation is that $\lim_{n\to\infty} \frac{S_\Delta}{S_{ABC}}$ and $\lim_{n\to\infty} \frac{S_{ABC}}{S_\Delta}$, in Table 1, are exact rational numbers for even values of k. This fact can be proved by the *index-reduction formula* (see (B. Johnson, 2003) and (R. C., Johnson, 2003)):

$$f_a f_b - f_c f_d = (-1)^r (f_{a-r} f_{b-r} - f_{c-r} f_{d-r})$$

which holds for arbitrary integers $a, b, c, d,\ r$ with $a + b = c + d$.

Indeed, substituting $a = 2k, b = k - 1, c = k, d = 2k - 1$ and $r = k - 1$ we get the following reduction:

$$f_{2k} f_{k-1} - f_k f_{2k-1} = (-1)^{k-1} (f_{k+1} f_0 - f_1 f_k) = (-1)^k f_k.$$

Hence, for even k, the following relation is satisfied:

$$\frac{f_{2k}}{f_k} = \frac{f_{2k-1} + 1}{f_{k-1}} = \alpha_k.$$

Substituting back in Eq. 2, we get the rational limit:

$$\lim_{n\to\infty} \frac{S_\Delta}{S_{ABC}} = 1 - \frac{3}{\alpha_k + 1}.$$

For $k = 2$, we have $\alpha_2 = \frac{f_4}{f_2} = 3$. Therefore, $\lim_{n\to\infty} \frac{S_\Delta}{S_{ABC}} = \frac{1}{4}$. Also, for $k = 4$, we have $\alpha_4 = \frac{f_8}{f_4} = 7$, and therefore $\lim_{n\to\infty} \frac{S_\Delta}{S_{ABC}} = \frac{5}{8}$. These results coincide with the appropriate values in Table 1.

Summarizing the above results we have the following corollary:

Corollary2 *If the sides BC, CA, AB of a triangle ABC are divided at L, M, N in the ratio $f_{n+k} : f_n$, where $n \geq 1$, $k \geq 1$, the cevians AL, BM, CN form a triangle Δ such that:*

$$\lim_{n\to\infty} \frac{S_\Delta}{S_{ABC}} = 1 - 3 \frac{f_k \varphi + f_{k-1}}{(f_{2k} + f_k)\varphi + f_{2k-1} + f_{k-1} + 1}.$$

For even k, the limit $\lim_{n\to\infty} \frac{S_{ABC}}{S_\Delta}$ is rational, and among all values of k, $k = 4$ gives the best possible approximation of $\frac{S_{ABC}}{S_\Delta}$ to the golden ratio.

Area Ratio of a Hexagon and a Triangle

Nice expressions of area ratios succeed in "symmetric" divisions. In particular the following theorem is proved in (Abboud, 2015):

Theorem2 *Supposethe sides of a triangle* $A_2A_3A_1$ *are divided at* $A_{i,1}, A_{i,2}$, $1 \le i \le 3$, *in the respective ratios* $1 : \lambda : 1$. *If the division points* $A_{i,1}, A_{i,2}$, $1 \le i \le 3$, *are connected to the opposite vertices then the ratio of the area of the resulting hexagon to the area of the triangle is*

$$\frac{2\lambda^2}{(3+\lambda)(2\lambda+3)}.$$

Special cases

1. If $\lambda = 1$ then the ratio of the area of the hexagon to the area of the triangle equals $\frac{1}{10}$. This special case is referred to as Marion Walter's theorem (Walter, 1993) which states the following: *If the trisection points of the sides of any triangle are connected to the opposite vertices, the resulting hexagon has one-tenth the area of the original triangle.*

2. If n is odd, $n = 2k+1$, then taking $\lambda = \frac{1}{k}$ implies that the ratio of the area of the hexagon to the area of the triangle is

$$\frac{2}{(3k+1)(3k+2)} = \frac{8}{9n^2-1}.$$

This special case is referred to as Morgan's theorem, which was proved by T. Watanabe, R. Hanson and F. D. Nowosielski (Watanabe et al., 1996) using Routh-Steiner theorem several times.

For encouraging my students to rediscover Morgan's formula we created the first two columns of Table 2, where r denotes the ratio of the area of the whole triangle to the area of the hexagon. The values of r in the second column were computed after constructing suitable figures using Geogebra. Then I asked the students to add a third column for $8r$ and a fourth column for $8r+1$. They noticed that all the numbers in the fourth column were multiplies of 9, so we added a new column for $\frac{8r+1}{9}$. For their surprise, they got the familiar sequence of perfect squares (column five).

Table 2: The numbers in the last column are perfect squares

n	r	$8r$	$8r+1$	$(8r+1)/9$
1	1	8	9	1
3	10	80	81	9
5	28	224	225	25
7	55	440	441	49
9	91	728	729	81

Thus, for odd n, we concluded with the relation

$$\frac{8r+1}{9}=n^2,$$

which is equivalent to the formula

$$r=\frac{9n^2-1}{8}.$$

3. If n is even, then taking $\lambda = n-2$ implies the following result (Abboud 2015):
If we divide each side of a given triangle into n equal parts, where n is even, thenthe ratio of the area of the hexagon, formed by connecting the cevians to the far most division points on the opposite sides, to the area of the triangle is

$$\frac{2(n-2)^2}{(n+1)(2n-1)}.$$

Table 3 shows some values.

Table 3: Values of area ratios for even *n*

n	$\frac{2(n-2)^2}{(n+1)(2n-1)}$
2	0
4	8/35
6	32/77
8	8/15
10	128/209

If $n=2$ the ratio is 0, since then the cevians intersect at one point. If n tends to infinity then, $\lim_{n\to\infty}\frac{2(n-2)^2}{(n+1)(2n-1)}=1$ which means that the inner triangle tends to cover the whole triangle.

New patterns

In the following we shall give a new activity to discover more patterns. Suppose the sides of a triangle $A_2A_3A_1$ are dividedat $A_{i,1}, A_{i,2}$, $1 \le i \le 3$, in the respective ratios $1 : \lambda : 2$, $\lambda > 0$ (see Fig. 2). Let I, J, K, L, M, N be the points of intersection of the corresponding cevians as shown in the table 4:

Table 4: Cevians and Points of intersection

point	cevians
I	$A_1A_{1,1} \cap A_3A_{3,2}$
J	$A_2A_{2,1} \cap A_3A_{3,2}$
K	$A_2A_{2,1} \cap A_1A_{1,2}$
L	$A_3A_{3,1} \cap A_1A_{1,2}$
M	$A_2A_{2,2} \cap A_3A_{3,1}$
N	$A_1A_{1,1} \cap A_2A_{2,2}$

Let G be the center of gravity of $A_1A_2A_3$ (see Fig. 2)$\cdot$ Since the barycentric coordinates of A_1, A_2, A_3 are $(1,0,0),(0,1,0),(0,0,1)$ respectively, the barycentric coordinates of G are $(\frac{1}{3},\frac{1}{3},\frac{1}{3})$. By symmetry, the area of the hexagon $IJKLMN$ is 3 times the area of the quadrilateral $GLMN$, which is the union of the triangles GMN and GLM (see Fig. 2). Therefore, it is sufficient to compute the barycentric coordinates of L, M and N.
Thus, to find the area ratio of the hexagon $IJKLMN$ to the triangle $A_2A_3A_1$ we should follow the following steps.

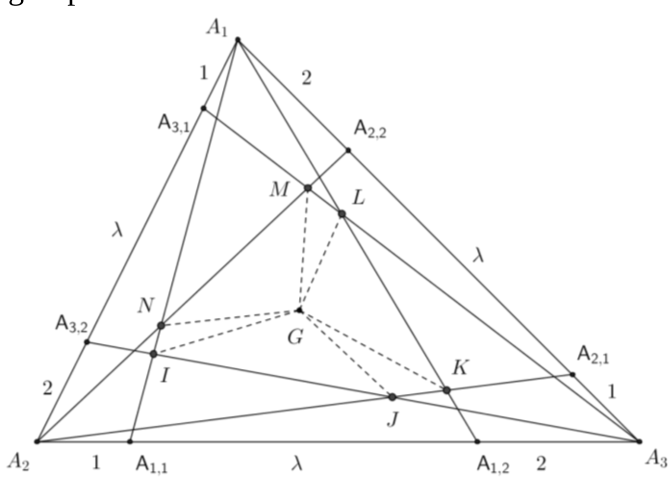

Figure 2: The sides of the triangle are divided in the ratios $1 : \lambda : 2$

Step 1 Find the barycentric coordinates of $A_{i,1}$ and $A_{i,2}$, $1 \le i \le 3$.
The points $A_{i,1}$, $1 \le i \le 3$ divide the sides A_2A_3, A_3A_1, A_1A_2 by the ratio $1 : \lambda + 2$, respectively, and the points $A_{i,2}, 1 \le i \le 3$, divide the sides A_2A_3, A_3A_1, A_1A_2 by the ratio $1 + \lambda : 2$, respectively. Hence, Table 5 shows the respective barycentric coordinates.

Table 5: The respective barycentric coordinates

point	barycentric coordinates
$A_{1,1}$	$(0,\lambda+2,1)$
$A_{2,1}$	$(1,0,\lambda+2)$
$A_{3,1}$	$(\lambda+2,1,0)$
$A_{1,2}$	$(0,2,\lambda+1)$
$A_{2,2}$	$(\lambda+1,0,2)$
$A_{3,2}$	$(2,\lambda+1,0)$

Step 2 Compute the equations of the cevians $A_1A_{1,1}, A_1A_{1,2}, A_2A_{2,2}$ and $A_3A_{3,1}$.
These equations can be computed as follows. The cevian $A_1A_{1,1}$ has the linear homogeneous equation

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & \lambda+2 & 1 \\ t_1 & t_2 & t_3 \end{vmatrix} = 0.$$

Computing this determinant we get $-t_2+(\lambda+2)t_3=0$. The cevian $A_1A_{1,2}$ has the equation

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & \lambda+1 \\ t_1 & t_2 & t_3 \end{vmatrix} = 0.$$

Equivalently, $-(\lambda+1)t_2+2t_3=0$. The cevian $A_2A_{2,2}$ has the equation

$$\begin{vmatrix} 0 & 1 & 0 \\ \lambda+1 & 0 & 2 \\ t_1 & t_2 & t_3 \end{vmatrix} = 0.$$

After computing the determinant we get the equation $-2t_1+(\lambda+1)t_3=0$. Finally, the cevian $A_3A_{3,1}$ has the equation

$$\begin{vmatrix} 0 & 0 & 1 \\ \lambda+2 & 1 & 0 \\ t_1 & t_2 & t_3 \end{vmatrix} = 0,$$

andthis simplifies into the equation $-t_1+(\lambda+2)t_2=0$.

Step 3 Find the barycentric coordinates of the points L, M and N.
Since, L is the intersection of the cevians $A_3A_{3,1} \cap A_1A_{1,2}$ then we have to solve the system of equations

$$\begin{cases} -t_1+(\lambda+2)t_2=0 \\ -(\lambda+1)t_2+2t_3=0 \end{cases}.$$

Substituting $t_2=2,$ we get $t_1=2(\lambda+2)$ and $t_3=\lambda+1$. Hence, $L=(2(\lambda+2),2,\lambda+1)$.
Likewise, M is the intersection of the cevians $A_2A_{2,2}\cap A_3A_{3,1}$. Therefore, we have to solve the following system of two equations:

$$\begin{cases}-2t_1+(\lambda+1)t_3=0\\ -t_1+(\lambda+2)t_2=0\end{cases}.$$

Substituting $t_1=(\lambda+1)(\lambda+2)$ we get $t_3=2(\lambda+2)$ and $t_2=\lambda+1$. Hence, $M=((\lambda+1)(\lambda+2),\lambda+1,2(\lambda+2))$.
Similarly, N is the intersection of the cevians $A_1A_{1,1}\cap A_2A_{2,2}$. Therefore, we have to solve the following system of two equations:

$$\begin{cases}-t_2+(\lambda+2)t_3=0\\ -2t_1+(\lambda+1)t_3=0\end{cases}.$$

Substituting $t_3=2$ we get $t_2=2(\lambda+2)$ and $t_1=(\lambda+1)$. Hence, $N=(\lambda+1,2(\lambda+2),2)$.

Step 4 Compute the areas of triangles GMN, GLM and normalize by dividing each determinant by the product of the sums of the rows.
The areas of GMN and GLM are proportional to the respective determinants

$$\begin{vmatrix}1/3 & 1/3 & 1/3\\ (\lambda+1)(\lambda+2) & \lambda+1 & 2(\lambda+2)\\ \lambda+1 & 2(\lambda+2) & 2\end{vmatrix}=\frac{2}{3}\lambda^3+\frac{5}{3}\lambda^2-\frac{7}{3}$$

and

$$\begin{vmatrix}1/3 & 1/3 & 1/3\\ 2(\lambda+2) & 2 & \lambda+1\\ (\lambda+1)(\lambda+2) & \lambda+1 & 2(\lambda+2)\end{vmatrix}=\frac{1}{3}\lambda^3-\frac{1}{3}\lambda^2-3\lambda-\frac{7}{3}.$$

Step 5 Find the ratio of the area of the hexagon $IJKLMN$ to the area of the triangle $A_2A_3A_1$.
We have to normalize the barycentric coordinates by dividing each one of the determinants by the product of the sums of the rows. Hence, the area of GMN is

$$\frac{\frac{2}{3}\lambda^3+\frac{5}{3}\lambda^2-\frac{7}{3}}{(\lambda^2+6\lambda+7)(3\lambda+7)},$$

times the area of the triangle $A_2A_3A_1$. Besides, the area of GLM is

$$\frac{\frac{1}{3}\lambda^3-\frac{1}{3}\lambda^2-3\lambda-\frac{7}{3}}{(\lambda^2+6\lambda+7)(3\lambda+7)},$$

times the area of the triangle $A_2A_3A_1$.
Therefore the area of the hexagon $IGKLMN$ equals

$$F(\lambda)=\frac{3\lambda^3+4\lambda^2-9\lambda-14}{(\lambda^2+6\lambda+7)(3\lambda+7)}$$

times the area of the triangle $A_2A_3A_1$.

Notice that, $\frac{1}{3}\lambda^3 - \frac{1}{3}\lambda^2 - 3\lambda - \frac{7}{3} = \frac{1}{3}(\lambda+1)(\lambda^2 - 2\lambda - 7)$. Hence, thispolynomial has one positive root: $\lambda = 2\sqrt{2}+1 \approx 3.8284$. In this case, the areas of the triangles GLM, GNI and GJK equal 0 and the hexagon degenerates to a triangle whose area is $F(2\sqrt{2}+1) = \frac{10}{7} - \frac{6}{7}\sqrt{2} \approx 0.21639$times the area of $A_2A_3A_1$ (see Fig. 3).

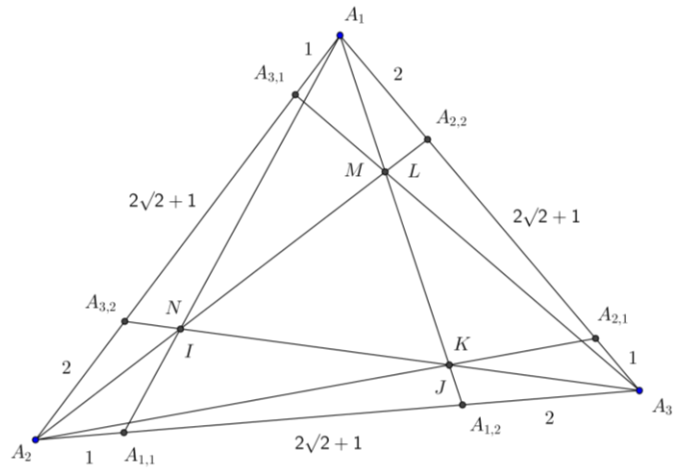

Figure 3: The hexagon degenerates to a triangle

Finally, summarizing the above results we have the following theorem:

Theorem3 *Suppose the sides of a triangle* $A_2A_3A_1$ *are divided at* $A_{i,1}, A_{i,2}$, $1 \le i \le 3$ *in the respective ratios* $1 : \lambda : 2$. *If the division points* $A_{i,1}, A_{i,2}$, $1 \le i \le 3$, *are connected to the opposite vertices then for* $\lambda \ge 2\sqrt{2}+1$ *the ratio of the area of the resulting hexagon to the area of the triangle is*

$$\frac{3\lambda^3 + 4\lambda^2 - 9\lambda - 14}{(\lambda^2 + 6\lambda + 7)(3\lambda + 7)}.$$

If $\lambda = 2\sqrt{2}+1$ *then the hexagon degenerates to a triangle whose area is* $\frac{10}{7} - \frac{6}{7}\sqrt{2}$ *times the area of* $A_2A_3A_1$.

References

E. Abboud (2015), On the Routh-Steiner theorem and some generalizations. The Mathematical Gazette, **99**, pp. 45-53.

H. S. M. Coxeter (1989), *Introduction to Geometry*, second edition,Wiley, New York.

R. C., Johnson (2003) "Fibonacci Identities by Matrix Methods and Generalisation to Related Sequences. http://maths.dur.ac.uk/~dma0rcj/PED/fib.pdf.

B. Johnson(2003), The Fibonacci Quarterly, Vol. **41**, B-960, p. 182.

J. Steiner (1882), Gesammelte Werke, Vol. 1, Reimer, Berlin.

E. J. Routh (1896), A Treatise on Analytical Statics with Numerous Examples, Vol. 1, second edition, Cambridge University Press.

M. Walter(1993), Reader Reflections: Marion's theorem, *Mathematics teacher* **86**,p. 619.
T. WatanabeR. Hanson and F. D. Nowosielski(1996),, Morgan's Theorem, *Mathematics teacher* **89**, 420-423.
D.Wells(1986), The Penguin Dictionary of Curious and Interesting Numbers, Middlesex, England: Penguin Books, pp. 61-67.