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Generalized Matrix T Distribution Through Generalized Multivariate Gamma Dristribution

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Abstract

In this paper, by conditioning the covariance structure of matrix variate normal distribution the construction of a generalized matrix t-type family is considered, thus providing a new perspective of this family. In this regard, a generalized multivariate gamma distribution including zonal polynomials is introduced. Some important statistical characteristics are given. An attempt is made to reconsider Bayes analysis of the column covariance matrix of the underlying population model.

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Introduction

In 1964, Lukacs and Laha defined the matrix/multi variate gamma (MG) distribution.

In multivariate statistical analysis, the MG distribution has been the subject of considerable interest, study, and applications for many years, since this distribution can be considered as the sample variance-covariance matrix when sampling from a weighted multivariate normal distribution. It also includes the well-known Wishart distribution as a special one. A systematic treatment of the Wishart and MG distributions can be found in Gupta and Nagar (2000). There is not much account of the MG distribution in the literature and neglected. Das and Dey (2007) considered some Bayesian issues of the MG distribution. Iranmanesh et al. (2010) considered this distribution in Bayesian analysis as a conjugate prior for the covariance matrix of a multivariate normal distribution. Iranmanesh et al. (2013) derived the inverted MG distribution and proposed many statistical characteristics along its usage in Bayesian context. Recently Nagar et al. (2013) defined an extended matrix variate gamma distribution by extending the multivariate gamma function due to Ingham and Siegel. In this paper, by making use of an integral relation containing zonal polynomials, we give a generalization to the result of Das and Dey (2007), which contains the latter as a special case.

In the following we provide the reader with the definition of the generalized multivariate gamma distribution.

Definition 1.1. A random matrix Y of order p is said to have a generalized multivariate gamma (GMG) distribution with parameters $\alpha, \beta, \kappa, \Sigma, U$ denoted by $Y \sim GMG_p(\alpha,\beta,\kappa,\Sigma,U)$, if its density is given by

$$f(Y) = \frac{\det(\Sigma)^{-\alpha}}{\beta^{\alpha p+k} C_\kappa(U\Sigma)\Gamma_p(\alpha,\kappa)} etr(-\frac{1}{\beta}\Sigma^{-1}Y)\det(Y)^{\alpha-(p+1)/2} C_\kappa(YU),\ Y>0 \qquad \textbf{(1)}$$

where $\alpha > \frac{(p-1)}{2} - k_p$, $\beta > 0$, $\Sigma > 0$, $U > 0$, $C_\kappa(.)$ is the zonal polynomial, $\kappa = (k_1,...,k_p)$ is a partition that $k_1 \geq ... \geq k_p$ and $(\alpha)_\kappa$ for a complex number is the generalized hypergeometric coefficent is defined

$$(\alpha)_\kappa = \prod_{i=1}^{m} \left(a - \frac{i-1}{2}\right)_{k_i} \qquad \textbf{(2)}$$

where $(a)_\kappa$ is a Pochhammer coefficent given by $(a)_k = a(a+1)...(a+k-1)$ and $\Gamma_p(\alpha,\kappa)$ is the generalized multivariate gamma function of weight κ, and $\Gamma_p(\alpha,\kappa) = \Gamma_p(\alpha)(\alpha)_\kappa$. (seeMuirhead, 2005).

The normalizing constant in Definition 1.1 obtained from the following

$$\int_{X>0} etr(-\mathrm{XZ})(\det \mathrm{X})^{\alpha-\frac{p+1}{2}} C_\kappa(\mathrm{XY})d\mathrm{X} = (\alpha)_\kappa \Gamma_p(\alpha)(\det \mathrm{Z})^{-\alpha} C_\kappa(\mathrm{YZ}^{-1}) \quad \textbf{(3)}$$

The density in (1) is a generalization of the matrix variate gamma distribution introduced by Das and Dey (2007). In the following result, the inverse GMG is defined.

Lemma 1.1. Let $X \sim GMG_p(\alpha, \beta, \kappa, \Sigma, U)$. Then $\mathrm{Y} = \mathrm{X}^{-1}$ has inverse GMG (IGMG) distribution denoted by $Y \sim IGMG_p(\alpha, \beta, \kappa, \Sigma, U)$ with the following density function

$$f(\mathrm{Y}) = \frac{\det(\Sigma)^{-\alpha}}{\beta^{\alpha p+k}\Gamma(\alpha,\kappa)_p C_\kappa(U\Sigma)} etr\left(-\frac{1}{\beta}\Sigma^{-1}\mathrm{Y}^{-1}\right)\det(\mathrm{Y})^{-\alpha-\frac{p+1}{2}} C_\kappa(\mathrm{Y}^{-1}U), \quad \mathrm{Y}>0 \quad \textbf{(4)}$$

Proof. The proof follows from the fact that the jacobian of transformation is given by $J(\mathrm{X} \to \mathrm{Y}) = \det(\mathrm{Y})^{-(p+1)}$.

For the case $\kappa = (0)$ since $C_\kappa \equiv 1$ and $(\alpha)_\kappa \equiv 1$, the distribution (4) reduces to the inverted matrix gamma distribution proposed by Iranmanesh et al. (2013). Alongside, we use the following definition.

Definition 1.2. The random matrix $\mathrm{X}(n \times p)$ is said to have a matrix variate normal distribution with mean $M(n \times p)$ and covariance matrix $\Omega \otimes \Sigma$ where $\Sigma(p \times p) > 0$ and $\Omega(n \times n) > 0$, if $vec(X) \sim N_{pn}(vec(M), \Omega \otimes \Sigma)$. We shall use the notation $X \sim N_{n,p}(M, \Omega \otimes \Sigma)$. The probability density function (p.d.f) of X is given by (Gupta and Nagar, 2000)

$$f(X) = (2\pi)^{-np/2}\det(\Omega)^{-p/2}\det(\Sigma)^{-n/2}$$
$$\times \exp\left[-\frac{1}{2}tr(\Omega^{-1}(\mathrm{X}-M)\Sigma^{-1}(\mathrm{X}-M)')\right], \qquad \mathrm{X} \in R^{n\times p}, \qquad \mathrm{M} \in R^{n\times p},$$

where $\otimes$ is the Kronecker product and vec is the vectorizing operation for matrix notation.

If $X \sim N_{n,p}(M, \Omega \otimes \Sigma)$, then the characteristic function of X is

$$\phi_X(Z) = E[\exp(tr(i\mathrm{Z}'X))]$$
$$= \exp\left[tr(i\mathrm{Z}'M - \frac{1}{2}\mathrm{Z}'\Omega\mathrm{Z}\Sigma)\right], \quad Z \in R^{n\times p},$$

Properties of GMG and IGMG Families

In this section, various properties of the GMG and IGMG distributions are derived.

Theorem 2.1. Let $\mathrm{Y} \sim GMG_p(\alpha,\beta,\kappa,\Sigma,U)$. Then the laplace transformation of Y is

$$\varphi_{\mathrm{Y}}(\mathrm{T}) = \frac{C_\kappa(U(\beta\mathrm{T}+\Sigma^{-1})^{-1})}{C_\kappa(U\Sigma)}\det(I_p+\Sigma\beta\mathrm{T})^{-\alpha} \qquad \textbf{(5)}$$

where T is a $p\times p$ matrix.

Proof. Let $C = \dfrac{\det(\Sigma)^{-\alpha}}{\beta^{\alpha p+k}\Gamma_p(\alpha,\kappa)C_\kappa(U\Sigma)}$, by definition we have

$$\begin{aligned}
\varphi_{\mathrm{Y}}(\mathrm{T}) &= E[\exp(-tr(\mathrm{TY}))],\\
&= \int_{\mathrm{Y}>0} etr(-\mathrm{TY})f(\mathrm{Y})d\mathrm{Y},\\
&= C\int_{\mathrm{Y}>0} etr(-\mathrm{TY})etr\left(-\frac{1}{\beta}\Sigma^{-1}\mathrm{Y}\right)\det(\mathrm{Y})^{\alpha-\frac{p+1}{2}}C_\kappa(\mathrm{Y}U)d\mathrm{Y},\\
&= C\int_{\mathrm{Y}>0} etr\left(-\mathrm{Y}\left(\mathrm{T}+\frac{1}{\beta}\Sigma^{-1}\right)\right)\det(\mathrm{Y})^{\alpha-\frac{p+1}{2}}C_\kappa(\mathrm{Y}U)d\mathrm{Y},
\end{aligned}$$

according to **(3)**

$$\begin{aligned}
\varphi_{\mathrm{Y}}(\mathrm{T}) &= \mathrm{C}\,\Gamma_p(\alpha,\kappa)\det\left(\mathrm{T}+\frac{1}{\beta}\Sigma^{-1}\right)^{-\alpha}C_\kappa\left(U\left(\mathrm{T}+\frac{1}{\beta}\Sigma^{-1}\right)^{-1}\right),\\
&= \frac{\det(\Sigma)^{-\alpha}\Gamma_p(\alpha,\kappa)\beta^{\alpha p}}{\Gamma_p(\alpha,\kappa)\beta^{\alpha p+k}C_\kappa(U\Sigma)}\det(\beta\mathrm{T}+\Sigma^{-1})^{-\alpha}C_\kappa(U\beta(\beta\mathrm{T}+\Sigma^{-1})^{-1}),\\
&= \frac{C_\kappa(U(\beta\mathrm{T}+\Sigma^{-1})^{-1})}{C_\kappa(U\Sigma)}\det(\mathrm{I}_p+\beta\Sigma\mathrm{T})^{-\alpha}.
\end{aligned}$$

Corollary 2.1.1. Let $\mathrm{Y} \sim GMG_p(\alpha,\beta,\kappa,\Sigma,U)$. Then the characteristic function of Y is

$$\psi_{\mathrm{Y}}(\mathrm{T}) = \frac{C_\kappa(U\Sigma(i\beta\Sigma\mathrm{T}-I_p)^{-1})}{C_\kappa(U\Sigma)}\det(i\beta\Sigma T-\mathrm{I}_p)^{-\alpha}. \quad \textbf{(6)}$$

Theorem 2.2. Let $\mathrm{Y} \sim GMG_p(\alpha,\beta,\kappa,\Sigma,U)$. Then

$$\mathrm{E}(\det(\mathrm{Y})^{\mathrm{h}}) = \frac{\Gamma_p(\alpha+h,\kappa)\beta^{\mathrm{hp}}\det(\Sigma)^{\mathrm{h}}}{\Gamma_p(\alpha,\kappa)}$$

Proof

$$E(\det(\mathrm{Y})^{h}) = \int_{\mathrm{Y}>0} \det(\mathrm{Y})^{h} f(\mathrm{Y})d\mathrm{Y},$$

$$= C\int_{\mathrm{Y}>0} etr\left(-\frac{1}{\beta}\Sigma^{-1}\mathrm{Y}\right)\det(\mathrm{Y})^{(\alpha+h)-\frac{p+1}{2}} C_\kappa(\mathrm{Y}U)d\mathrm{Y},$$

$$= \frac{\det(\Sigma)^{-\alpha}}{\beta^{\alpha p+k}C_\kappa(U\Sigma)\Gamma_p(\alpha,\kappa)}\Gamma_p(\alpha+h,\kappa)\beta^{((\alpha+h)p+k)}\det(\Sigma)^{\alpha+h}C_\kappa(U\Sigma),$$

$$= \frac{\Gamma_p(\alpha+h,\kappa)\det(\Sigma)^{h}\beta^{hp}}{\Gamma_p(\alpha,\kappa)}.$$

in order to find the expectation of the trace of an GMG random variable, it is useful to find the expectation of zonal polynomial, which is stated below.

Theorem 2.3. Let $\mathrm{X}(p\times p) \sim GMG_p(\alpha,\beta,\kappa,\Sigma,U)$ and $\mathrm{B}(p\times p)$ be a constant symmetric matrix. Then

$$E(C_\tau(\mathrm{XB})) = C\beta^{p\alpha+t+k}\det(\Sigma)^{\alpha}\sum_{\phi\in\kappa,\tau}\theta_\phi^{\kappa,\tau}\Gamma_p(\alpha,\phi)C_\phi^{\kappa,\tau}(U\Sigma,\mathrm{B}\Sigma),$$

where $\Gamma_p(\alpha,\phi) = (\alpha)_\phi\Gamma_p(\alpha)$.

proof. By definition we have

$$E(C_\tau(\mathrm{XB})) = \int_{\mathrm{X}>0} C_\tau(\mathrm{XB})f(\mathrm{X})d\mathrm{X},$$

$$= C\sum_{\phi\in\kappa,\tau}\theta_\phi^{\kappa,\tau}\int_{\mathrm{X}>0} etr\left(-\frac{1}{\beta}\Sigma^{-1}\mathrm{X}\right)\det(\mathrm{X})^{\alpha-\frac{p+1}{2}} C_\kappa(\mathrm{X}U)C_\tau(\mathrm{XB})d\mathrm{X},$$

using the fact that (Eq. (2.8) of Davis, 1979)

$$C_\kappa(\mathrm{X}U)C_\tau(\mathrm{XB}) = \sum_{\phi\in\kappa,\tau}\theta_\phi^{\kappa,\tau}C_\phi^{\kappa,\tau}(\mathrm{X}U,\mathrm{XB})$$

we obtain

$$E(C_\tau(\mathrm{XB})) = C\sum_{\phi\in\kappa,\tau}\theta_\phi^{\kappa,\tau}\int_{\mathrm{X}>0} etr\left(-\frac{1}{\beta}\Sigma^{-1}\mathrm{X}\right)\det(\mathrm{X})^{\alpha-\frac{p+1}{2}}C_\phi^{\kappa,\tau}(\mathrm{X}U,\mathrm{XB})d\mathrm{X},$$

$$= C\sum_{\phi\in\kappa,\tau}\theta_\phi^{\kappa,\tau}\Gamma_p(\alpha,\phi)\det\left(\frac{1}{\beta}\Sigma^{-1}\right)^{-\alpha}C_\phi^{\kappa,\tau}(\beta U\Sigma,\beta\mathrm{B}\Sigma),$$

the result follows from the fact that
$C_\phi^{\kappa,\tau}(\beta U\Sigma,\beta\mathrm{B}\Sigma) = \beta^{k+t}C_\phi^{\kappa,\tau}(U\Sigma,\mathrm{B}\Sigma).$

Theorem 2.4. Let $\mathrm{Y} \sim IGMG_p(\alpha,\beta,\kappa,\Sigma,U)$. Then the Laplace transformation of Y

is given by

$$\varphi_{\mathrm{Y}}(\mathrm{T}) = \frac{1}{[\Gamma_p(\alpha,\kappa)]\beta^{\alpha p+k}\det(\Sigma)^\alpha C_\kappa(U\Sigma)}\mathrm{B}(U,\Sigma,\mathrm{T};\alpha,\beta,\kappa), \quad \textbf{(7)}$$

where

$$\mathrm{B}(U,\Sigma,\mathrm{T};\alpha,\beta,\kappa) = \int_{\mathrm{X}>0}\det(\mathrm{X})^{\alpha-\frac{p+1}{2}}C_\kappa(\mathrm{X}U)\times B_0\left(\frac{1}{\beta}\Sigma^{-1}(\mathrm{T}+\mathrm{X})\right)d\mathrm{X}, \quad \textbf{(8)}$$

and $B_0(.)$ is the Bessel function of matrix argument (see Herz(1955) is given by

$$B_\delta(W\mathrm{Z}) = \det(W)^{-\delta}\int_{S>0}\det(S)^{\delta-\frac{p+1}{2}}etr(-S\mathrm{Z}-S^{-1}W)dS.$$

proof.

$$\begin{aligned}\varphi_{\mathrm{Y}}(\mathrm{T}) &= E(etr(-\mathrm{TY})),\\ &= \int_{\mathrm{Y}>0} etr(-\mathrm{TY})\, f(\mathrm{Y})d\mathrm{Y},\\ &= C\int_{\mathrm{Y}>0} etr(-\mathrm{TY})etr\left(-\frac{1}{\beta}\Sigma^{-1}\mathrm{Y}^{-1}\right)\det(\mathrm{Y})^{-\alpha-\frac{p+1}{2}} C_\kappa(\mathrm{Y}^{-1}U)d\mathrm{Y},\\ &= C\int_{\mathrm{Y}>0} etr\left(-\left(\mathrm{TY}+\frac{1}{\beta}\Sigma^{-1}\mathrm{Y}^{-1}\right)\right)\det(\mathrm{Y})^{-\alpha-\frac{p+1}{2}} C_\kappa(\mathrm{Y}^{-1}U)d\mathrm{Y},\end{aligned}$$

(**9**) since Y and U are symmetric matrices, then the eigenvalues of $U\mathrm{Y}^{-1}, \mathrm{Y}^{-1}U$ are equal, so $C_\kappa(U\mathrm{Y}^{-1}) = C_\kappa(\mathrm{Y}^{-1}U)$. According to (**3**)

$$C_\kappa(\mathrm{Y}^{-1}U) = \frac{1}{\Gamma_p(\alpha,\kappa)\det(\mathrm{Y})^{-\alpha}}\int_{\mathrm{X}>0} etr(-\mathrm{XY})\det(\mathrm{X})^{\alpha-\frac{p+1}{2}} C_\kappa(\mathrm{X}U)d\mathrm{X} \quad \textbf{(10)}$$

Substituting (**9**) in (**10**) and change the order of integration we have

$$\begin{aligned}\varphi_{\mathrm{Y}}(\mathrm{T}) = C\det(\mathrm{Y})^{\alpha-\frac{p+1}{2}} etr\left(-\left(\mathrm{TY}+\frac{1}{\beta}\Sigma^{-1}\mathrm{Y}^{-1}\right)\right)\Bigg(\frac{1}{\Gamma_p(\alpha,\kappa)\det(\mathrm{Y})^{-\alpha}}\int_{\mathrm{X}>0} etr(-\mathrm{XY})\det(\mathrm{X})^{\alpha-\frac{p+1}{2}}\\ \times\int_{\mathrm{X}>0} etr(-\mathrm{XY})\det(\mathrm{X})^{\alpha-\frac{p+1}{2}} C_\kappa(\mathrm{X}U)d\mathrm{X}\Bigg)d\mathrm{Y},\end{aligned}$$

$$\begin{aligned}&= \frac{C}{\Gamma_p(\alpha,\kappa)}\int_{\mathrm{X}>0}\det(\mathrm{X})^{\alpha-\frac{p+1}{2}} C_\kappa(\mathrm{X}U)\Bigg(\int_{\mathrm{Y}>0}\det(\mathrm{Y})^{-\frac{p+1}{2}}\\ &\qquad\times etr\left(-\left((\mathrm{T}+\mathrm{X})\mathrm{Y}+\frac{1}{\beta}\Sigma^{-1}\mathrm{Y}^{-1}\right)\right)d\mathrm{Y}\Bigg)d\mathrm{X},\\ &= C^*\int_{\mathrm{X}>0}\det(\mathrm{X})^{\alpha-\frac{p+1}{2}} C_\kappa(\mathrm{X}U)B_0\left(\frac{1}{\beta}\Sigma^{-1}(\mathrm{T}+\mathrm{X})\right)d\mathrm{X},\\ &= C^*\mathrm{B}(U,\Sigma,\mathrm{T};\alpha,\beta,\kappa),\end{aligned}$$

where

$$C^* = \frac{1}{[\Gamma_p(\alpha,\kappa)]^2\beta^{\alpha p+k}\det(\Sigma)^{\alpha} C_\kappa(U\Sigma)}$$

and $B_\delta(.)$ is the Bessel function of matrix argument.

Theorem 2.5. Let $\mathrm{Y} \sim IGMG_p(\alpha,\beta,\kappa,\Sigma,U)$. Then

$$E[\det(\mathrm{Y})^h] = \frac{\Gamma_p(\alpha - h, \kappa)}{\Gamma_p(\alpha, \kappa)\det(\Sigma)^h \beta^{hp}}$$

proof.

$$E[\det(\mathrm{Y})^h] = \int_{\mathrm{Y}>0} \det(\mathrm{Y})^h f(\mathrm{Y})d\mathrm{Y},$$

$$= C\int_{\mathrm{Y}>0} etr\left(-\frac{1}{\beta}\Sigma^{-1}\mathrm{Y}^{-1}\right)\det(\mathrm{Y})^{-(\alpha-h)-\frac{p+1}{2}} C_\kappa(\mathrm{Y}^{-1}U)d\mathrm{Y},$$

where we used the fact that

$$\int_{\mathrm{Y}>0} etr\left(-\frac{1}{\beta}\Sigma^{-1}\mathrm{Y}^{-1}\right)\det(\mathrm{Y})^{-\alpha-\frac{p+1}{2}} C_\kappa(\mathrm{Y}^{-1}U)d\mathrm{Y} = \beta^{\alpha p+k}\Gamma_p(\alpha,\kappa)C_\kappa(U\Sigma)\det(\Sigma)^\alpha.$$

so

$$E[\det(\mathrm{Y})^h] = \frac{\det(\Sigma)^{-\alpha}}{\beta^{\alpha p+k}\Gamma_p(\alpha,\kappa)C_\kappa(U\Sigma)}\beta^{(\alpha-h)p+k}\Gamma_p(\alpha-h,\kappa)C_\kappa(U\Sigma)\det(\Sigma)^{\alpha-h},$$

$$= \frac{\det(\Sigma)^{-h}\beta^{-hp}\Gamma_p(\alpha-h,\kappa)}{\Gamma_p(\alpha,\kappa)},$$

$$= \frac{\Gamma_p(\alpha-h,\kappa)}{\Gamma_p(\alpha,\kappa)\det(\Sigma)^h\beta^{hp}}.$$

Theorem 2.6. Let $W \sim IGMG_p(\alpha,\beta,\kappa,\Sigma,U)$ and $\mathrm{A}(p\times p)$ be a constant symmetric matrix. Then

$$\mathrm{A}W\mathrm{A}' \sim IGMG_p(\alpha,\beta,\kappa,\mathrm{A}'^{-1}\Sigma\mathrm{A}^{-1},\mathrm{A}'U\mathrm{A})$$

proof. The jacobian of transformation $\mathrm{Y} = \mathrm{A}W\mathrm{A}'$ is $J(W\to\mathrm{Y}) = \det(\mathrm{A})^{-(p+1)}$, the density of Y is

$$f(\mathrm{Y}) = Cetr\left(-\frac{1}{\beta}\Sigma^{-1}(\mathrm{A}^{-1}\mathrm{Y}(\mathrm{A}')^{-1})^{-1}\right)\det(\mathrm{A}^{-1}\mathrm{Y}(\mathrm{A}')^{-1})^{-\alpha-\frac{p+1}{2}} C_\kappa((\mathrm{A}^{-1}\mathrm{Y}(\mathrm{A}')^{-1})U)\det(\mathrm{A})^{-(p+1)},$$

$$= Cetr\left(-\frac{1}{\beta}\Sigma^{-1}\mathrm{A}\mathrm{Y}^{-1}\mathrm{A}'\right)\det(\mathrm{A}^{-1})^{-\alpha-\frac{p+1}{2}}\det((\mathrm{A}')^{-1})^{-\alpha-\frac{p+1}{2}}\det(\mathrm{Y})^{-\alpha-\frac{p+1}{2}} C_\kappa\left(\mathrm{A}\mathrm{Y}^{-1}\mathrm{A}'U\right)$$

$$\times\det(\mathrm{A})^{-(p+1)},$$

$$= \det(\mathrm{A})^{\alpha+\frac{p+1}{2}}\det(\mathrm{A}')^{\alpha+\frac{p+1}{2}}\det(\mathrm{Y})^{-\alpha-\frac{p+1}{2}} C_\kappa\left(\mathrm{A}\mathrm{Y}^{-1}\mathrm{A}'U\right)\det(\mathrm{A})^{-(p+1)},$$

$$= Cetr\left(-\frac{1}{\beta}\Sigma^{-1}\mathrm{A}\mathrm{Y}^{-1}\mathrm{A}'\right)\det(\mathrm{A})^{\alpha}\det(\mathrm{A}')^{\alpha} C_\kappa\left(\mathrm{A}\mathrm{Y}^{-1}\mathrm{A}'U\right)\det(\mathrm{Y})^{-\alpha-\frac{p+1}{2}},$$

$$= \frac{\det((\mathrm{A}')^{-1}\Sigma\mathrm{A}^{-1})^{-\alpha}}{\beta^{\alpha p+k}\Gamma_p(\alpha,\kappa)C_\kappa(U\Sigma)} etr\left(-\frac{1}{\beta}((\mathrm{A}')^{-1}\Sigma\mathrm{A}^{-1})^{-1}\mathrm{Y}^{-1}\right)\det(\mathrm{Y})^{-\alpha-\frac{p+1}{2}} C_\kappa\left(\mathrm{A}\mathrm{Y}^{-1}\mathrm{A}'U\right),$$

because A is symmetric the eigen values of $\mathrm{A}\mathrm{Y}^{-1}\mathrm{A}'U$ and $\mathrm{Y}^{-1}\mathrm{A}'U\mathrm{A}$ are equal,

$$C_\kappa(\mathrm{A}\mathrm{Y}^{-1}\mathrm{A}'U) = C_\kappa(\mathrm{Y}^{-1}\mathrm{A}'U\mathrm{A}).$$

Then

$$f(\mathrm{Y}) = \frac{\det((\mathrm{A}')^{-1}\Sigma\mathrm{A}^{-1})^{-\alpha}}{\beta^{\alpha p+k}\Gamma_p(\alpha,\kappa)C_\kappa(U\Sigma)} etr\left(-\frac{1}{\beta}((\mathrm{A}')^{-1}\Sigma\mathrm{A}^{-1})^{-1}\mathrm{Y}^{-1}\right)\det(\mathrm{Y})^{-\alpha-\frac{p+1}{2}} C_\kappa(\mathrm{Y}^{-1}\mathrm{A}'U\mathrm{A}).$$

Theorem 2.7. Let $\mathrm{Y} \sim IGMG_p(\alpha,\beta,\kappa,\Sigma,U)$ and $\mathrm{A}(p\times p)$ be a constant symmetric matrix and $\mathrm{X} = \mathrm{A}\mathrm{Y}$. Then the pdf of X is given by

$$f(\mathrm{X}) = \frac{\det(\mathrm{A}^{-1}\Sigma)^{-\alpha}}{\beta^{\alpha p+k}\Gamma_p(\alpha,\kappa)C_\kappa(U\Sigma)} etr\left(-\frac{1}{\beta}(\mathrm{A}^{-1}\Sigma)^{-1}\mathrm{X}^{-1}\right)\det(\mathrm{A})^{-\frac{p+1}{2}}\det(\mathrm{X})^{-\alpha-\frac{p+1}{2}} C_\kappa(\mathrm{X}^{-1}\mathrm{A}U),$$

proof. by $J(\mathrm{X}\to\mathrm{Y}) = \det(\mathrm{A})^{-p}$, we have

$$f_\mathrm{X}(\mathrm{X}) = f_\mathrm{Y}(\mathrm{A}^{-1}\mathrm{X})J(\mathrm{X}\to\mathrm{Y}),$$

$$= Cetr\left(-\frac{1}{\beta}\Sigma^{-1}\mathrm{X}^{-1}\mathrm{A}\right)\det(\mathrm{A}^{-1}\mathrm{X})^{-\alpha-\frac{p+1}{2}} C_\kappa(\mathrm{X}^{-1}\mathrm{A}U)\det(\mathrm{A})^{-p},$$

$$= \frac{\det(\mathrm{A}^{-1}\Sigma)^{-\alpha}}{\beta^{\alpha p+k}\Gamma_p(\alpha,\kappa)C_\kappa(U\Sigma)} etr\left(-\frac{1}{\beta}(\mathrm{A}^{-1}\Sigma)^{-1}\mathrm{X}^{-1}\right)\det(\mathrm{A})^{-\frac{p+1}{2}}\det(\mathrm{X})^{-\alpha-\frac{p+1}{2}} C_\kappa(\mathrm{X}^{-1}\mathrm{A}U)$$

Theorem 2.8. Let $Y_1 \sim GMG_p(\alpha_1,\beta,\kappa,\Sigma,U)$ and $Y_2 \sim GMG_p(\alpha_2,\beta,\kappa,\Sigma,U)$, and
Y_1, Y_2 be independent. Then the distribution of $W = Y_1(Y_1+Y_2)^{-1}$ is

$$f(W) = C\beta^{p(\alpha_1+\alpha_2+\frac{-p+1}{2})+k+t}\det(\Sigma)^{\alpha_1+\alpha_2+\frac{-p+1}{2}}\sum_{\phi\in\kappa.\tau}\theta_\phi^{\kappa,\tau}\det(W)^{\alpha_1-\frac{p+1}{2}}\det(I_p-W)^{\alpha_2-\frac{p+1}{2}}$$
$$\times\Gamma_p\left(\alpha_1+\alpha_2+\frac{-p+1}{2},\phi\right)C_\phi^{\kappa,\tau}(\Sigma UW,\Sigma(I_p-W)U)$$

proof. If $W = Y_1(Y_1+Y_2)^{-1}$ and $V = Y_1+Y_2$, Then $Y_1 = WV$ and $Y_2 = V(I_p - W)$. The jacobian of transformation is

$$J(Y_1, Y_2 \to W, V) = Y_1 + Y_2.$$

The joint distribution of

$$f(Y_1,Y_2) = f(Y_1)f(Y_2),$$
$$= \frac{1}{\Gamma_p(\alpha_2,\kappa)\beta^{\alpha_1 p+k}C_\kappa(U\Sigma)}etr\left(-\frac{1}{\beta}\Sigma^{-1}Y_1\right)\det(Y_1)^{\alpha_1-\frac{p+1}{2}}C_\kappa(Y_1U)$$
$$\times\frac{1}{\Gamma_p(\alpha_2,\kappa)\beta^{\alpha_2 p+k}C_\tau(U\Sigma)}etr\left(-\frac{1}{\beta}\Sigma^{-1}Y_2\right)\det(Y_2)^{\alpha_2-\frac{p+1}{2}}C_\tau(Y_2U),$$
$$= \frac{1}{\Gamma_p(\alpha_1,\kappa)\Gamma_p(\alpha_2,\kappa)\beta^{p(\alpha_1+\alpha_2)+k}C_\kappa(U\Sigma)C_\tau(U\Sigma)}etr\left(-\frac{1}{\beta}\Sigma^{-1}(Y_1+Y_2)\right)$$
$$\times\det(Y_1)^{\alpha_1-\frac{p+1}{2}}\det(Y_2)^{\alpha_2-\frac{p+1}{2}}C_\kappa(Y_1U)C_\tau(Y_2U).$$

The joint distribution of V and W is given by

$$f(W,V) = f(WV, V(I_p-W))J(Y_1,Y_2\to W, V = Y_1+Y_2),$$
$$= \frac{1}{\Gamma_p(\alpha_1,\kappa)\Gamma_p(\alpha_2,\kappa)\beta^{p(\alpha_1+\alpha_2)+k}C_\kappa(U\Sigma)C_\tau(U\Sigma)}etr\left(-\frac{1}{\beta}\Sigma^{-1}V\right)\det(WV)^{\alpha_1-\frac{1}{2}(P+1)}$$
$$\times\det(V(I_p-W))^{\alpha_2-\frac{1}{2}(P+1)}C_\kappa(WVU)C_\kappa(V(I_p-W)U)\det(V),$$
$$= \frac{\Gamma_p(\alpha_1+\alpha_2,\kappa)}{\Gamma_p(\alpha_1,\kappa)\Gamma_p(\alpha_2,\kappa)}\frac{1}{\beta^{(\alpha_1+\alpha_2)p+k}C_\kappa(U\Sigma)C_\tau(U\Sigma)\Gamma_p(\alpha_1+\alpha_2,\kappa)}etr\left(-\frac{1}{\beta}\Sigma^{-1}V\right)$$
$$\times\det(W)^{\alpha_1-\frac{p+1}{2}}\det(I_p-W)^{\alpha_2-\frac{p+1}{2}}\det(V)^{\alpha_1+\alpha_2-p}C_\kappa(WVU)C_\tau(V(I_p-W)U).$$

By integrating over V we have

$$f(W)=\int_{V>0} f(W,V)dV,$$

$$=C\det(W)^{\alpha_1-\frac{p+1}{2}}\det(\mathrm{I}_p-W)^{\alpha_2-\frac{p+1}{2}}\int_{V>0}\det(V)^{\alpha_1+\alpha_2-p}etr\left(-\frac{1}{\beta}\Sigma^{-1}V\right)C_\kappa(WVU)$$

$$\times C_\tau(V(\mathrm{I}_p-W)U)dV,$$

$$=C\sum_{\phi\in\kappa,\tau}\theta_\phi^{\kappa,\tau}\det(W)^{\alpha_1-\frac{p+1}{2}}\det(\mathrm{I}_p-W)^{\alpha_2-\frac{p+1}{2}}\int_{V>0}\det(V)^{\alpha_1+\alpha_2+\frac{-p+1}{2}-\frac{p+1}{2}}etr\left(-\frac{1}{\beta}\Sigma^{-1}V\right)$$

$$\times C_\phi^{\kappa,\tau}(VUW,V(\mathrm{I}_p-W)U)dV,$$

$$=C\beta^{p(\alpha_1+\alpha_2+\frac{-p+1}{2})+k+t}\det(\Sigma)^{\alpha_1+\alpha_2+\frac{-p+1}{2}}\sum_{\phi\in\kappa,\tau}\phi_\phi^{\kappa,\tau}\det(W)^{\alpha_1-\frac{p+1}{2}}\det(\mathrm{I}_p-W)^{\alpha_2-\frac{p+1}{2}}$$

$$\times\Gamma_p\left(\alpha_1+\alpha_2+\frac{-p+1}{2},\phi\right)C_\phi^{\kappa,\tau}(\Sigma UW,\Sigma(\mathrm{I}_p-W)U).$$

Family of Generalized Matrix t-Distributions

In this section, the main result of the paper concerning the construction of the new family of matrix variate t-distributions is presented. This distribution will be applied in the Bayesian context.

Definition 3.1. The random matrix $\mathrm{T}(n\times p)$ is said to have a generalized matrix variate t-distribution (GMT) with parameters $M\in R^{n\times p},\Psi(p\times p)>0,\Omega(n\times n)>0,\ U(p\times p)>0,$ $\alpha>p-1/2,\beta>0,\kappa=(k_1,...,k_p),k_1\geq...\geq k_p$ if its p.d.f is given by

$$f(\mathrm{T})=\frac{\det(\Psi)^{n/2}\det(\Omega)^{-p/2}\Gamma_p(\alpha+n/2,\kappa)}{\pi^{np/2}\Gamma_p(\alpha,\kappa)C_\kappa(U\Psi)}\left(\frac{2}{\beta}\right)^{-np/2+k}C_\kappa\left(U\left(T'\Omega^{-1}T+\frac{2}{\beta}\Psi^{-1}\right)^{-1}\right)$$

$$\times\det\left[\mathrm{I}_n+\frac{\beta}{2}\Omega^{-1}(\mathrm{T}-M)\Psi(\mathrm{T}-M)'\right]^{-(\alpha+n/2)} \qquad \textbf{(11)}$$

we shall use the notation $\mathrm{T}\sim GMT_{n,p}(\alpha,\beta,\kappa,M,\Omega,\Psi,U)$.

For $\beta = 2, \alpha = \frac{n+p-1}{2}$ and $k = 0$ then $\kappa = (0)$, the GMT distribution simplifies to the matrix T distribution with n degrees of freedom. (See Gupta and Nagar(2000))

Further for the case $\kappa = (0)$, the GMT simplifies to the generalized t-distribution defined by Iranmanesh et al. (2010).

Theorem 3.1. Let $\mathrm{X}|\Sigma \sim N_{n,p}(0, \Omega \otimes \Sigma)$ and $\Sigma \sim IGMG_p(\alpha, \beta, \kappa, \Psi, U)$. Then, $\mathrm{X} \sim GMT_{n,p}(\alpha, \beta, \kappa, 0, \Omega, \Psi, U)$.

proof. Using conditional method, we find

$$f(\mathrm{X}) = \int_{\Sigma>0} g(\mathrm{X}|\Sigma) h(\Sigma) d\Sigma,$$

$$= \frac{\det(\Psi)^{-\alpha} \det(\Omega)^{-p/2}}{\Gamma_p(\alpha,\kappa)(2\pi)^{np/2}\beta^{\alpha p+k} C_\kappa(U\Psi)} \int_{\Sigma>0} \det(\Sigma)^{-(\alpha+(n+p+1)/2)}$$

$$\times \exp\left[-\frac{1}{\beta} tr\left(\left(\frac{\beta}{2}\mathrm{X}'\Omega^{-1}\mathrm{X} + \Psi^{-1}\right)\Sigma^{-1}\right)\right] C_\kappa(\Sigma^{-1}U) d\Sigma,$$

$$= \frac{\beta^{np/2}(2\pi)^{-np/2}\det(\Omega)^{-p/2}\det(\Psi)^{-\alpha}\Gamma_p(\alpha + n/2, \kappa)}{\Gamma_p(\alpha,\kappa) C_\kappa(U\Psi)}$$

$$\times \quad C_\kappa\left[U\left(\frac{\beta}{2}\mathrm{X}'\Omega^{-1}\mathrm{X} + \Psi^{-1}\right)^{-1}\right] \det\left(\frac{\beta}{2}\mathrm{X}'\Omega^{-1}\mathrm{X} + \Psi^{-1}\right)^{-(\alpha+n/2)}$$

(12)

where the (12) is derived from (6). Now we used the fact that

$$C_\kappa\left[U\left(\frac{\beta}{2}\mathrm{X}'\Omega^{-1}\mathrm{X} + \Psi^{-1}\right)^{-1}\right] = C_\kappa\left[U\left(\frac{\beta}{2}\left(\mathrm{X}'\Omega^{-1}\mathrm{X} + \frac{2}{\beta}\Psi^{-1}\right)\right)^{-1}\right]$$

$$= \left(\frac{2}{\beta}\right)^k C_\kappa\left[U\left(\mathrm{X}'\Omega^{-1}\mathrm{X} + \frac{2}{\beta}\Psi^{-1}\right)^{-1}\right].$$

and from the following identity

$$\det\left(\Psi^{-1} + \frac{\beta}{2}\mathrm{X}'\Omega^{-1}\mathrm{X}\right) = \det(\Psi^{-1})\det\left(\mathrm{I}_p + \frac{\beta}{2}\Psi\mathrm{X}'\Omega^{-1}\mathrm{X}\right)$$

$$= \det(\Psi^{-1})\det\left(\mathrm{I}_n + \frac{\beta}{2}\Omega^{-1}\mathrm{X}\Psi\mathrm{X}'\right),$$

Finally we obtain

$$f(\mathrm{X}) = \frac{\Gamma_p(\alpha + n/2, \kappa)(2/\beta)^{-np/2+k} \det(\Omega)^{-p/2} \det(\Psi)^{n/2}}{\Gamma_p(\alpha,\kappa) C_\kappa(U\Psi)\pi^{np/2}}$$
$$\times C_\kappa\left[U\left(\mathrm{X}'\Omega^{-1}\mathrm{X} + \frac{2}{\beta}\Psi^{-1}\right)^{-1}\right]\det\left(\mathrm{I}_n + \frac{\beta}{2}\Omega^{-1}\mathrm{X}\Psi\mathrm{X}'\right)^{-(\alpha+n/2)}$$

Some Properties of the GMT Family

In this section, various properties of the GMT distribution are studied using its p.d.f.

Theorem 4.1. If $\mathrm{T} \sim GMT_{n,p}(\alpha,\beta,\kappa,M,\Omega,\Psi,U)$,
then $\mathrm{T}' \sim GMT_{n,p}(\alpha,\beta,\kappa,M',\Omega,\Psi,U)$.
proof. By noting that

$$\det\left(\mathrm{I}_n + \frac{\beta}{2}\Omega^{-1}(\mathrm{T}-M)\Psi(\mathrm{T}-M)'\right) = \det\left(\mathrm{I}_p + \frac{\beta}{2}\Psi^{-1}(\mathrm{T}'-M')\Omega(\mathrm{T}'-M')'\right),$$

the result follows. (See page 137 Gupta and Nagar, 2000})

Theorem 4.2. Let $\mathrm{T} \sim GMT_{n,p}(\alpha,\beta,\kappa,M,\Omega,\Psi,U)$ and $\mathrm{A}(n\times n)$ and $\mathrm{B}(p\times p)$ be nonsingular matrices, then
$\mathrm{ATB} \sim GMT_{n,p}(\alpha,\beta,\kappa,\mathrm{A}M\mathrm{B},\mathrm{A}\Omega\mathrm{A}',\mathrm{B}^{-1}\Psi(\mathrm{B}')^{-1},\mathrm{B}'U\mathrm{B})$.
proof. Transforming $W = \mathrm{ATB}$, with the Jacobian of transformation
$J(\mathrm{T}\to W) = \det(\mathrm{A})^{-p}\det(\mathrm{B})^{-n}$, from density (**11**) of T follows the density of W equals

$$
\begin{aligned}
f(W) &= f_{\mathrm{T}}(\mathrm{A}^{-1}W\mathrm{B}^{-1})J(\mathrm{T}\to W),\\
&=\frac{\Gamma_p(\alpha+n/2,\kappa)(2/\beta)^{-np/2+k}\det(\Omega)^{-p/2}\det(\Psi)^{n/2}}{\Gamma_p(\alpha,\kappa)C_\kappa(U\Psi)\pi^{np/2}}\det(\mathrm{A})^{-p}\det(\mathrm{B})^{-n}\\
&\quad\times C_\kappa\left[U\left((\mathrm{A}^{-1}W\mathrm{B}^{-1})'\Omega^{-1}(\mathrm{A}^{-1}W\mathrm{B}^{-1})+\frac{2}{\beta}\Psi^{-1}\right)^{-1}\right]\\
&\quad\times\det\left(\mathrm{I}_n+\frac{\beta}{2}\Omega^{-1}(\mathrm{A}^{-1}W\mathrm{B}^{-1}-M)\Psi(\mathrm{A}^{-1}W\mathrm{B}^{-1}-M)'\right)^{-(\alpha+n/2)}\\
&=\frac{\Gamma_p(\alpha+n/2,\kappa)}{\Gamma_p(\alpha,\kappa)(2\pi/\beta)^{np/2}C_\kappa(U\Psi)}\det(\mathrm{A}\Omega\mathrm{A}')^{-p/2}\det(\mathrm{B}^{-1}\Psi(\mathrm{B}^{-1})')^{n/2}\\
&\quad\times C_\kappa\left[U\left((\mathrm{B}^{-1})'W'(\mathrm{A}^{-1})'\Omega^{-1}\mathrm{A}^{-1}W\mathrm{B}^{-1}+\frac{2}{\beta}\Psi^{-1}\right)^{-1}\right]
\end{aligned}
$$

$$
\begin{aligned}
&\quad\times\det\left(\mathrm{I}_n+\frac{\beta}{2}\Omega^{-1}\mathrm{A}^{-1}(W-\mathrm{A}M\mathrm{B})\mathrm{B}^{-1}\Psi(\mathrm{B}^{-1})'(W-\mathrm{A}M\mathrm{B})'(\mathrm{A}^{-1})'\right)^{-(\alpha+n/2)}\\
&=\frac{\Gamma_p(\alpha+n/2,\kappa)}{\Gamma_p(\alpha,\kappa)(2\pi/\beta)^{np/2}C_\kappa(U\Psi)}\det(\mathrm{A}\Omega\mathrm{A}')^{-p/2}\det(\mathrm{B}^{-1}\Psi(\mathrm{B}^{-1})')^{n/2}\\
&\quad\times C_\kappa\left[\mathrm{B}'U\mathrm{B}\left(W'(\mathrm{A}^{-1})'\Omega^{-1}\mathrm{A}^{-1}W+\frac{2}{\beta}\mathrm{B}'\Psi\mathrm{B}^{-1}\right)^{-1}\right]\\
&\quad\times\det\left(\mathrm{I}_n+\frac{\beta}{2}(\mathrm{A}\Omega\mathrm{A}')^{-1}(W-\mathrm{A}M\mathrm{B})(\mathrm{B}'\Psi^{-1}\mathrm{B})^{-1}(W-\mathrm{A}M\mathrm{B})'\right)^{-(\alpha+n/2)},
\end{aligned}
$$

where $W\in R^{n\times p}$ and, hence the result. □

Theorem 4.3. Let $\mathrm{T}\sim GMT_{n,p}(\alpha,\beta,\kappa,M,\Omega,\Psi),$ then the characteristic function of T is given by

$$
\phi_{\mathrm{T}}(\mathrm{Z})=\frac{\exp[tr(i\mathrm{Z}'M)]\det(\Psi)^{-\alpha}}{[\Gamma_p(\alpha,\kappa)]^2\beta^{\alpha p+k}C_\kappa(U\Psi)}\mathrm{B}(U,\Psi,\frac{1}{2}\mathrm{Z}'\Omega\mathrm{Z},\alpha,\beta,\kappa).
$$

where $\mathrm{B}_\delta(.)$ is defied in (**8**).

proof.

$$\begin{aligned}\phi_T(\mathrm{Z}) &= E[\exp(tr(i\mathrm{Z}'\mathrm{T}))], \qquad \mathrm{Z} \in R^{n\times p}\\ &= E[E(\exp(tr(i\mathrm{Z}'\mathrm{T}|\Sigma)))],\\ &= E[\exp(tr(i\mathrm{Z}'M - \frac{1}{2}\mathrm{Z}'\Omega\mathrm{Z}\Sigma))],\\ &= e^{tr(i\mathrm{Z}'M)}E[(-\frac{1}{2}\mathrm{Z}'\Omega\mathrm{Z}\Sigma)]\end{aligned}$$

Since $\Sigma \sim IGMG_p(\alpha,\beta,\kappa,\Psi,U)$, from (**7**) we have

$$\begin{aligned}\phi_{\mathrm{T}}(\mathrm{Z}) &= \frac{\exp[tr(i\mathrm{Z}'M)]\det(\Psi)^{-\alpha}}{[\Gamma_p(\alpha,\kappa)]^2\beta^{\alpha p+k}C_\kappa(U\Psi)}\int_{\mathrm{X}>0}\det(\mathrm{X})^{\alpha-(p+1)/2}C_\kappa(\mathrm{X}U)\\ &\quad\times \mathrm{B}_0\left(\frac{1}{\beta}\Psi^{-1}\left(\frac{1}{2}\mathrm{Z}'\Omega\mathrm{Z}+\mathrm{X}\right)\right)d\mathrm{X}\\ &= \frac{\exp[tr(i\mathrm{Z}'M)]\det(\Psi)^{-\alpha}}{[\Gamma_p(\alpha,\kappa)]^2\beta^{\alpha p+k}C_\kappa(U\Psi)}\mathrm{B}(U,\Psi,\frac{1}{2}\mathrm{Z}'\Omega\mathrm{Z},\alpha,\beta,\kappa).\end{aligned}$$

where $\mathrm{B}_0(.)$ is the type two Bessel function of matrix argument.

In the following section another point of view is considered concerning the result obtained in Theorem (**3.1**) This result is an extension to the work of Dickey(1967).

Theorem 4.4. let $\mathrm{X} \sim N_{n,p}(0,\mathrm{I}_n \otimes \Omega)$, independent of $S \sim GMG_n(\alpha,\beta,\kappa,\Lambda^{-1},U)$ define

$$\mathrm{T} = (S^{-1/2})'\mathrm{X} + M,$$

where $M(n\times p)$ is a constant matrix and $S^{1/2}(S^{1/2})' = S$.
Then, $\mathrm{T} \sim GMT_{n,p}(\alpha,\beta,\kappa,M,\Lambda^{-1},\Omega,U)$.

proof. The joint density of X and S is given by

$$\begin{aligned}f(\mathrm{X},S) &= \frac{(2\pi)^{-np/2}\det(\Omega)^{-n/2}\det(\Lambda)^{\alpha}}{\beta^{n\alpha+k}\Gamma_n(\alpha,\kappa)C_\kappa(U\Lambda^{-1})}\det(S)^{\alpha-(n+1)/2}C_\kappa(SU)\\ &\quad\times\exp\left[-tr\left(\frac{1}{\beta}\Lambda S+\frac{1}{2}\mathrm{X}\Omega^{-1}\mathrm{X}'\right)\right],\end{aligned}$$

where $S>0, \mathrm{X}\in R^{n\times p}$. Now, let $\mathrm{T}=(S^{-1/2})'\mathrm{X}+M$. The Jacobian of transformation is
$J(\mathrm{X}\to\mathrm{T})=\det(S)^{p/2}$. Substituting for X in terms of T in the joint density of X and S, and multiplying the resulting expression by $J(\mathrm{X}\to\mathrm{T})$, we get the joint p.d.f of T and S as

$$f(\mathrm{T},S)=\frac{(2\pi)^{-np/2}\det(\Omega)^{-n/2}\det(\Lambda)^{\alpha}}{\beta^{n\alpha+k}\Gamma_n(\alpha,\kappa)C_\kappa(U\Lambda^{-1})}\det(S)^{\alpha-(n+1)/2+p/2}C_\kappa(SU)$$
$$\times\exp\left[-tr\left(\frac{1}{\beta}\left(\Lambda+\frac{\beta}{2}S^{1/2}(\mathrm{T}-M)\Omega^{-1}(\mathrm{T}-M)'(S^{1/2})'\right)\right)\right]'$$
$$=\frac{(2\pi)^{-np/2}\det(\Omega)^{-n/2}\det(\Lambda)^{\alpha}}{\beta^{n\alpha+k}\Gamma_n(\alpha,\kappa)C_\kappa(U\Lambda^{-1})}\det(S)^{\alpha-(n+1)/2+p/2}C_\kappa(SU)$$
$$\times\exp\left[-tr\left(\frac{1}{\beta}\left(\Lambda+\frac{\beta}{2}(\mathrm{T}-M)\Omega^{-1}(\mathrm{T}-M)'\right)\right)S\right],$$

where $S>0, \mathrm{T}\in R^{n\times p}$.

Now integrating out S and by using (3) the density of T is obtained as

$$f(\mathrm{T})=\frac{(2\pi)^{-np/2}\det(\Omega)^{-n/2}\det(\Lambda)^{\alpha}}{\beta^{n\alpha+k}\Gamma_n(\alpha,\kappa)C_\kappa(U\Lambda^{-1})}\int_{S>0}\det(S)^{\alpha-(n+1)/2+p/2}C_\kappa(SU)$$
$$\times\exp\left[-tr\left(\frac{1}{\beta}\left(\Lambda+\frac{\beta}{2}(\mathrm{T}-M)\Omega^{-1}(\mathrm{T}-M)'\right)\right)S\right]dS,$$

$$= \frac{(2\pi)^{-np/2}\det(\Omega)^{-n/2}\det(\Lambda)^{\alpha}\beta^{np/2}\Gamma_n(\alpha+p/2,\kappa)}{\Gamma_n(\alpha,\kappa)C_\kappa(U\Lambda^{-1})}$$
$$\times\left(\frac{2}{\beta}\right)^k C_\kappa\left[U\left(\frac{2}{\beta}\Lambda+(\mathrm{T}-M)\Omega^{-1}(\mathrm{T}-M)'\right)^{-1}\right]$$
$$\times\det(\Lambda)^{-(\alpha+p/2)}\det\left[\mathrm{I}_n+\frac{\beta}{2}\Lambda^{-1}(\mathrm{T}-M)\Omega^{-1}(\mathrm{T}-M)'\right]^{-(\alpha+p/2)},$$
$$= \frac{\det(\Omega)^{-n/2}\det(\Lambda)^{-p/2}\Gamma_n(\alpha+p/2,\kappa)}{\Gamma_n(\alpha,\kappa)C_\kappa(U\Lambda^{-1})(\pi)^{np/2}}\left(\frac{2}{\beta}\right)^{k-np/2}$$
$$\times C_\kappa\left[U\left(\frac{2}{\beta}\Lambda+(\mathrm{T}-M)\Omega^{-1}(\mathrm{T}-M)'\right)^{-1}\right]$$
$$\times\det\left[\mathrm{I}_n+\frac{\beta}{2}\Lambda^{-1}(\mathrm{T}-M)\Omega^{-1}(\mathrm{T}-M)'\right]^{-(\alpha+p/2)}$$

Bayes Estimation

For application purposes, in this section the Bayes estimator of Σ based on the conditional property is derived. In this regard, we consider Kullback Leibler divergence loss (KLDL) as the measurement. First we state a result due to Das and Dey(2010).

Lemma 5.1. Suppose A is an estimator for unknown parameters matrix Σ, where $\pi(\mathrm{A}|D)$ and $\pi(\Sigma|D)$ are the corresponding posterior probability density function over R^p respectively, where D indicates Data. Now the posterior expected loss of A, when the posterior distribution is $\pi(\Sigma|D)$, is

$$\rho(\Sigma,\mathrm{A}) = \int_{\Sigma>0} L(\Sigma,\mathrm{A})dF^{\pi(\Sigma|D)}d\Sigma,$$

where the loss function is $\log\left(\frac{\pi(\mathrm{A}|D)}{\pi(\mathrm{A}|\Sigma)}\right)$. Then the Bayes rule is

$$\delta^{\pi}(D) = \arg\max_{\Sigma>0}\pi(\Sigma|D),$$

which is the mode of posterior distribution of $\pi(\Sigma|D)$.

Note that the posterior expected loss function in Lemma 5.1,

$$\rho(\Sigma,\mathrm{A}) = E\left[\log\left(\frac{\pi(\mathrm{A}|D)}{\pi(\Sigma|D)}\right)\right],$$

can be interpreted as the Kullback Leibler divergence of the posterior distribution evaluated under action A from the true posterior distribution of unknown parameters Σ. Therefore the posterior expected loss or Kullback Leibler divergence is minimum, if we choose our action as posterior mode.

Lemma 5.2. Let $\mathrm{X}|\Sigma \sim N_{n,p}(0,\Omega\otimes\Sigma)$. Further Σ has prior distribution as $IGMG_p(\alpha,\beta,\kappa,\Psi,U)$. Then the posterior distribution of Σ is

$$\pi(\Sigma|\mathrm{X}) = \frac{\det(\Psi)^{-(\alpha+n/2)}\det(\Sigma)^{-(\alpha+n/2+(p+1)/2)}C_\kappa(\Sigma^{-1}U)}{2^k\Gamma_p(\alpha+n/2,\kappa)\beta^{p\alpha+np/2}C_\kappa\left[U\left(\mathrm{X}'\Omega^{-1}\mathrm{X}+\frac{2}{\beta}\Psi^{-1}\right)^{-1}\right]}$$

$$\times\det\left(\mathrm{I}_n+\frac{\beta}{2}\Omega^{-1}\mathrm{X}\Psi^{-1}\mathrm{X}'\right)^{\alpha+n/2} etr\left(-\frac{1}{2}\Omega^{-1}\mathrm{X}\Sigma^{-1}\mathrm{X}'-\frac{1}{\beta}\Psi^{-1}\Sigma^{-1}\right).$$

Proof. By definition

$$\pi(\Sigma|\mathrm{X}) = \frac{f(\mathrm{X}|\Sigma)\pi(\Sigma)}{m(\mathrm{X})}.$$

By applying Theorem (3.1) we obtain the underlying result.

According to Lemma 5.2 the Bayes estimator of Σ under KLDL function is given by

$$\hat{\Sigma} = \arg\max_\Sigma \pi(\Sigma|\mathrm{X}).$$

Iranmanesh et al.(2010) showed that

$$\hat{\Sigma} = [\alpha+n/2+(p+1)/2]^{-1}\left(\frac{1}{2}\mathrm{X}'\Omega^{-1}\mathrm{X}+\frac{1}{\beta}\Psi\right),$$

for the especial case k=0.

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