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Assembling Large Entangled States in the Rényi-Ingarden-Urbanik Entropy Measure under the *SU*(2)-Dynamics Decomposition for Systems Built from Two-Level Subsystems

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This paper should be cited as follows:

Delgado Cepeda, F. J. (2019). **"Assembling Large Entangled States in the Rényi-Ingarden-Urbanik Entropy Measure under the *SU*(2)-Dynamics Decomposition for Systems Built from Two-Level Subsystems",** Athens: ATINER'S Conference Paper Series, No: **PHY2018-2621.**

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URL Conference Papers Series: www.atiner.gr/papers.htm

ISSN: 2241-2891
26/02/2019

Assembling Large Entangled States in the Rényi-Ingarden-Urbanik Entropy Measure under the *SU*(2)-Dynamics Decomposition for Systems Built from Two-Level Subsystems

Francisco Javier Delgado

Abstract

Quantum Information is a discipline derived from Quantum Mechanics which uses quantum systems to exploit their states as information recipients. Normally, these states are conformed by two-level systems to reproduce the binary nature underlying the classical computation structure. Quantum evolution is then controlled to reproduce convenient information processing operations. Evolution could be hard to be controlled. SU(2) decomposition procedure lets to set a binary structure of processing when a convenient basis is selected to set the dynamics description. In this work, we exploit this procedure for a generic Hamiltonian in order to set the process to reduce arbitrary states into simplest ones. For this work, we use customary SU(2) operations on local and entangled states. These operations are described in the development. They involve 1, 2 and 4-local operations meaning the number of quantum parties involved, in agreement with the decomposition procedure scope. This task is complex in spite the difficulty to set a general way to manipulate the entanglement in the system. We are particularly interested on the application of stochastic procedures based in SU(2) decomposition operations to achieve that goal. In order to have a measure of the advancement of the last task, we use the Rényi-Ingarden-Urbanik entropy to describe the whole spectrum of entanglement in the large systems through the assembling/disassembling of the state.

Keywords: Entanglement, Quantum Information, Quantum Processing, Rényi-Ingarden-Urbanik Entropy, States Design.

Acknowledgments: Support for research activity is acknowledged to CONACYT. Economic support to disseminate the current results through NOVUS grants initiative is acknowledged to Tecnológico de Monterrey.

Introduction

Quantum information is pursuing novel approaches to set information and processing on physical systems exhibiting quantum properties, such as superposition and entanglement. There, normally two-level systems are combined to scale them into large physical systems being able to hold large amounts of complex Information. Due to the entangling interactions among them, which are mandatory as part of the improved processing, these composed systems exhibit a complex dynamics hard to control under a universal set of operations as in classical computing, normally in the form of gates. In the most of cases, the whole evolution matrices for the composed system have their entries different from zero. This aspect does hard to fit them in controlled operations, normally involving two informational parts in the physical systems (remember that normally, those states are expressed in the eigenstates of the physical parameters composing such subsystems). Thus, the clue is to select an adequate basis to re-express the dynamics in order to reduce the complexity of such evolution. Nevertheless, the physical meaningful of those states are not assured, it is only informational unlinked from the physical parameters.

SU(2) decomposition (Delgado 2017a) is a procedure recently developed for certain architectures and interactions (three interactions and set-ups denoted by Type I, II_{ab} and III are able to set this kind of controllable operations; despite could be generalizable on a wider spectrum of quantum systems and non-only as *SU*(2) reduction, instead having another group structure). There, generic *SU*(2) operations can be settled in order to design states and to process information in terms of the Di Vincenzo criteria (Di Vincenzo 1997). Such procedure could be useful to reach processing gates in terms of the Gates Based Quantum Computation (GBQC), control in quantum information, entanglement quantification, quantum error correction, etc. In the current work, we exploit this approach to set key operations to transform random large quantum states in order to analyse if they are able to carry out the state through the entire spectrum of entanglement.

The aim of this paper is focused in the complexity to get general quantum states departing from the most simple ones, particularly those possibly exhibiting several types of entanglement (really, from separable to the genuine entangled states as a spectrum). The second section summarizes the details of *SU*(2) decomposition procedure. Third section depicts the general processing operations being possible to reach in such procedure. Fourth section sets several findings about entanglement already reached under the *SU*(2) decomposition operations. Fifth section presents the Rényi-Ingarden-Urbanik (RIU) entropy (Ingarden and Urbanik 1962) and finally sets several automated procedures to show the evolution processing of large quantum states transiting from separable to entangled states. Last section set the conclusions and future work.

SU(2)-Dynamics Decomposition

In this section, we set the generalities of *SU*(2) reduction procedure as it has been developed in (Delgado 2015), together with the Hamiltonian being considered in such development. This procedure was first applied in the Heisenberg-Ising Hamiltonian for two particle spin together with external magnetic fields in a fix direction. By using the Bell states basis to describe the dynamics, it was discovered that the *SU*(4) dynamics for the entire system became split in two evolution subspaces, each one with a *SU*(2) dynamics. This block structure in the evolution matrix was useful to set the most common quantum processing operations. After, it was realized this procedure could be generalizable to any system integrated by several interacting two level systems being combined in only one bigger system.

A General Hamiltonian in SU(2^{2d})

Thus, the current analysis of the *SU*(2) decomposition departs from the general Hamiltonian for $n=2d$ qubits (Delgado 2017a):

$$\tilde{H} = \sum_{\{i_k\}} h_{\{i_k\}} \bigotimes_{k=1}^{n} \sigma_{i_k} = \sum_{\mathcal{I}=0}^{4^n-1} h_{\mathcal{I}_4^n} \bigotimes_{k=1}^{n} \sigma_{\mathcal{I}_{4,k}^n} \tag{1}$$

where matrices σ_i , $i=0, 1, 2, 3$ are the Pauli matrices (and the identity), and symbol $\bigotimes$ is the tensor product. The two-biased notation in the subscripts of the Hamiltonian coefficients and Pauli matrices are common in quantum information: $\{i_1, i_2, \ldots, i_n\}$ are a set of subscripts as reference of each coefficient and each physical part of the system, but alternatively $\mathcal{I}$ is an equivalent number to those subscripts considered as its expression in base 4 with n digits. Thus, $\mathcal{I}^{n}_{4,k} = i_k$ is its k digit in such base. We switch both notations under convenience.

The Generalized Bell Basis (GBS)

The last expression is settled on the Hilbert space $\mathcal{H}^{2d}$ of spin states for each one of the $2d$ subsystems being combined, a space with 2^{2d} complex dimensions (or 4^{2d-1} parametric dimensions due to the normalization condition). Procedure suggests re-express the dynamics on the Generalized Bell States basis, GBS (Sych and Leuchs 2009):

$$\begin{aligned}\left|\Psi_{\mathcal{I}_4^d}\right\rangle &= \bigotimes_{s=1}^{d}\frac{1}{\sqrt{2}}\sum_{\epsilon_s,\delta_s=0}^{1}(\tilde{\sigma}_{i_s})_{\epsilon_s,\delta_s}\left|\epsilon_s\delta_s\right\rangle \\ &= \frac{1}{\sqrt{2^d}}\sum_{\{\epsilon_j\},\{\delta_k\}}(\tilde{\sigma}_{i_1}\otimes\ldots\otimes\tilde{\sigma}_{i_d})_{\epsilon_1\ldots\epsilon_d,\delta_1\ldots\delta_d}\left|\epsilon_1\ldots\epsilon_d\right\rangle\otimes\left|\delta_1\ldots\delta_d\right\rangle \\ &= \frac{1}{\sqrt{2^d}}\sum_{\mathcal{E},\mathcal{D}=0}^{2^d-1}(\tilde{\sigma}_{i_1}\otimes\ldots\otimes\tilde{\sigma}_{i_d})_{\mathcal{E}_2^d,\mathcal{D}_2^d}\left|\mathcal{E}_2^d\right\rangle\otimes\left|\mathcal{D}_2^d\right\rangle\end{aligned} \tag{2}$$

where the tilded σ_i are the modified Pauli matrices, differing from the classical ones only for i=2 by an additional imaginary unit factor, i. Scripts follow the same rule that in the Hamiltonian, as instance: $\mathcal{D}^n_{2,k} = \delta_k$, in this case in base 2. Then, the Hamiltonian components can be expressed (Delgado 2017a) as:

$$\left\langle\Psi_{\mathcal{I}_4^d}|H|\Psi_{\mathcal{K}_4^d}\right\rangle = \frac{1}{2^d}\sum_{\mathcal{J}=0}^{4^{2d}-1}h_{\mathcal{J}_4^{2d}}\prod_{s=1}^{d}\mathrm{Tr}(\tilde{\sigma}^*_{i_s}\sigma_{j_{d+s}}\tilde{\sigma}^T_{k_s}\sigma^T_{j_s}) \tag{3}$$

This expression sets the rules to get the *SU*(2) decomposition. By taking pairs of the entire set of scripts $\mathcal{I}^d_4$ and $\mathcal{K}^d_4$, in such way that in both corresponding rows of Hamiltonian only the diagonal terms and those with entries having them as subscripts, we will get a 2 × 2 block. In the current Hamiltonian, we can to get three possible groups of interactions: Types I, $\mathrm{II}_{a,b}$ and III (Delgado 2017a). All of them contain entangling operations between the pairs 1, d+1; 2, 4; …; d, 2d (called correspondent pairs), a kind of operations referred as 2-local operations. All of them exhibit Ising-like interactions between correspondent pairs providing the diagonal part of the blocks. Additionally, Ising-like interactions among the non-correspondent parts provide the diagonal-off entries in the blocks. Correspondingly, the Type II_b and III involve Dzyaloshinskii-Moriya-like interactions (Dzyaloshinskii 1958, Moriya 1960) among non-correspondent parts and correspondent pairs respectively to provide the diagonal-off entries (Delgado 2017a). In any case, all interactions work in only two ways: a) they lets to mix or to exchange the basis states by pairs if they are different in only one script (Type I ad III), or b) they are different in exactly two scripts (Types $\mathrm{II}_{a,b}$). These exchange rules have been described in detail by Delgado (2017b).

The Block Structure Generated by the SU(2) *Reduction*

Last decomposition lets to express the Hamiltonian in the form (by the rearrangement of the basis elements) of 2×2 blocks:

$$H = \bigotimes_{i=1}^{2^{n-1}}\mathbb{S}_{Hi} = \left(\begin{array}{c|c|c|c} \mathbb{S}_{H1} & \mathbf{0} & \ldots & \mathbf{0} \\ \hline \mathbf{0} & \mathbb{S}_{H2} & \ldots & \mathbf{0} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \mathbf{0} & \mathbf{0} & \ldots & \mathbb{S}_{H2^{n-1}} \end{array}\right) \tag{4}$$

Thus, because the evolution operator can be calculated from the Hamiltonian as the time-ordered integral (Grossman and Katz 1972), basically containing time-ordered products preserving that structure, it inherits the same block structure:

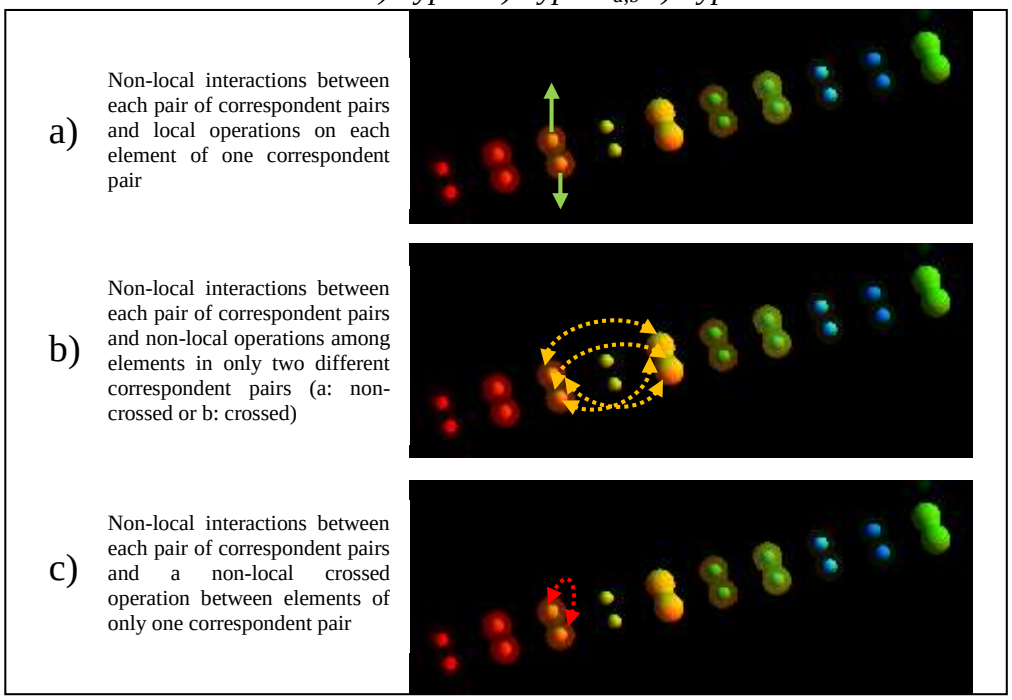
$$U = \bigotimes_{i=1}^{2^{n-1}} \mathbb{S}_{Ui} = \begin{pmatrix} \mathbb{S}_{U1} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbb{S}_{U2} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbb{S}_{U2^{n-1}} \end{pmatrix} \quad (5)$$

Clearly there are 2^{2d-1} blocks in each matrix with a respective phase factor (despite only 2^{2d-2} are independent, thus generating a semi product structure). Only U exhibits the structure $U(1)^{2^{2d-2}} \times SU(2)^{2^{2d-1}}$ (Delgado 2017a).

Interactions Generating SU(2) *Reduction*

Only the Type $II_{a,b}$ interactions are able to generate extended entanglement in the system because it works on two correspondent pairs (4-local operations), it means on states differing in two elements of the subscripts. Figure 1 depicts those interactions showing $2d$=20 qubits. The envelope surface depicts the permanent entanglement (in some variable degree) between the correspondent pairs maintained by non-local interactions in all three cases (non-crossed and Ising-like). Type I interaction includes only local interactions in each element of only one correspondent pair. Type III interaction includes non-local crossed (Dzyaloshinskii-Moriya-like) interactions between elements of only one correspondent pair. These interactions do not generate more extended entanglement far away than the pair where are applied. Finally, Type $II_{a,b}$ interactions include non-local, and non-crossed (a) or crossed (b) interactions able to generate extended entanglement far away from the correspondent pairs. As a result, under this scheme, we have a group of $SU(2)$ based operations through the whole matrix evolution to produce quantum information processing with well identified operations.

Figure 1. *Depiction of the Three Types of Interactions Exhibiting SU(2) Reduction in the GBS Basis a) Type I b) Type II$_{a,b}$ c) Type III*



Operations for Quantum Information and their Classification

This section is devoted to depict some generic operations in quantum processing. In spite of the *SU*(2) reduction, then each block is able to adopt this form. They are shown able to generate large entangled states departing from basic (separable) or similar quantum information states (as a re-scaling in the number of qubits). Still, the operations presented are not necessarily the most general, instead the most common operations in terms of similitude with those used as traditional gates in GBQC.

Generic Operations in Quantum Processing

A general common operation (but not the most general actually) in the previous *SU*(2) reduction context can be expressed as:

$$\mathbb{H}_m^c(\delta,\epsilon)_{\mathcal{I},\mathcal{I}'} \equiv (-1)^m \begin{pmatrix} \epsilon & i^c\delta \\ (-i)^c\delta & -\epsilon \end{pmatrix} \tag{6}$$

where $\varepsilon^2+\delta^2=1$; m, $c \in \mathbf{Z}$. In particular, we can identify there certain archetypical operations (Delgado, 2017b). By instance, Hadamard-like gate:

$$H_{\mathcal{I},\mathcal{I}'}^{m,c,s_\epsilon} = \frac{(-1)^m}{\sqrt{2}} \begin{pmatrix} s_\epsilon & i^c \\ (-i)^c & -s_\epsilon \end{pmatrix} \tag{7}$$

or the exchange gate, responsible to switch the states related in the block:

$$E^{m,c}_{\mathcal{I},\mathcal{I}'} = (-1)^m \begin{pmatrix} 0 & i^c \\ (-i)^c & 0 \end{pmatrix} \tag{8}$$

and the quasi-identity gate:

$$e^{i\alpha\pi}\mathbb{I}_{\mathcal{I},\mathcal{I}'} \equiv I^{\alpha}_{\mathcal{I},\mathcal{I}'} \tag{9}$$

where: $\mathbb{I}$ is the identity matrix, s_ε is a sign, $\alpha \in \mathbf{R}$. In all cases, $\mathcal{I}$, $\mathcal{I}'$ in $[0, 2^{2d}]$ depict the rows (or columns) where the block is allocated. Despite (6) is a more general gate under the *SU*(2) decomposition able to be adapted into other necessities. Delgado (2017b, 2017c) has presented several results about the use of these operations in order to get special extended entangled states as Greenberger–Horne–Zeilinger states (*GHZ*) and *W* states. Despite, they are very limited cases of large entangled states and they are easy because contain a reduced number of terms (despite of their importance). In the current approach, we are interested on the analysis of general processes to reach states with arbitrary degrees of entanglement. Normally, it has been probed that while size of the combined system becomes larger, the number of entangled states grows, thus a random state in $\mathcal{H}^{2d}$ becomes normally an entangled state, and particularly exhibiting a large value of entanglement (Enriquez et al. 2015).

Reaching General States with 1, 2 and 4-Local Operations

In this section, we describe some previous works developing the generation of maximal entangled states in the context of *SU*(2) reduction operations, despite limited because they are special cases, they have set probes about how basic processing operations are able to generate large entangled states. The aim of this section is to prepare the reader for a stochastic process to reduce (or alternatively to construct) arbitrary large quantum states into other simpler. Operations being presented should be responsible to assemble or to disassemble each state considered in each step of the process. Each operation has a group of parameters to be selected, we are interested in the best selection able to reduce or to increase the entanglement as a roadmap.

Processes Generating Entanglement under SU(2) *Reduction Procedures*

2-local operations let to the entanglement inside of the correspondent pairs but no longer that it. Particularly, in this context, the absolute local operations (the 1-local), are a special case of 2-local ones. While, 4-local operations generate entanglement involving more than two pairs. Note no longer exchange of entanglement is directly allowed at a time in the nature because physical

interactions are between pairs of physical particles. Because the *SU*(2) reduction sets the interaction architecture in the system and there the GBS basis works as a universal basis still if the three types of interactions are combined, then is few recommendable as first instance to change the structure of correspondent pairs because it will destroy the stable basis on which the dynamics is being analysed.

Particularly, by using the Partial Trace criterion for pure states (Nielsen and Chuang 2000) to measure the entanglement degree between two basis elements of the entire system:

$$|\phi_{\mathcal{I}\mathcal{J}}\rangle = \alpha_{\mathcal{I}}\,|\Psi_{\mathcal{I}}\rangle + \alpha_{\mathcal{J}}\,|\Psi_{\mathcal{J}}\rangle \tag{10}$$

Concretely, we are interested mainly on the effect of one of the 4-local operations exchanging the indexed $\mathcal{I}$ and $\mathcal{J}$ (assuming they are different in a couple of digits in base 4). Then, under such criterion, the concurrence is:

$$\mathcal{C}^2(\mathrm{Tr}^S(\rho_{\mathcal{I}\mathcal{J}})) \;=\; 2(1-\mathrm{Tr}^S(\rho^2_{\mathcal{I}\mathcal{J}})) \tag{11}$$

where *S* is the set of scripts left after to take the partial trace (Ulhmann 2000). We assume they are *k*, *k*''. If $m=\min(m_1, m_2)$, where m_1, m_2 are the dimensions of each part in the partial trace, then concurrence values ranges from 0 (for separable states) to $2(m\text{-}1)/m$ (maximal entangled states). For our case, $m=m_1=m_2=4$ because we are comparing the entanglement between one corresponding pair with another. By expressing conveniently the state coefficients as:

$$\alpha_{\mathcal{I}} = \cos\theta/2, \alpha_{\mathcal{J}} = e^{i\phi}\sin\theta/2 \tag{12}$$

Then, we get by computing the density matrix and then the concurrence C^2:

$$\tfrac{3}{2} - \tfrac{1}{2}\sin^2\theta(\cos^2\phi'\delta_{i_{k'}j_{k'}}\delta_{i_{k''}j_{k''}} + \sin^2\phi'(1-\delta_{i_{k'}j_{k'}}\delta_{i_{k''}j_{k''}})) \tag{13}$$

clearly denoting the possible maximum value 3/2. In this case:

$$\bar{\sigma}_i \equiv e^{i\phi_i}\sigma_i \tag{14}$$

$$\phi' = \phi + \phi_{i'_k} - \phi_{j'_k} \tag{15}$$

Nevertheless, this is important only for the case i=2 and those cases involving it (there, $\phi_i = \pi/2$). This result exhibits how entanglement is exchanged between two correspondent pairs under the 4-local operation generating the linear combination (10). This fact appoints on the generation of extended entanglement, which can be continued if interactions consider now

other non-correspondent pairs. This aspect will be fundamental in this work because it is the source of extended entanglement.

Several cases have been developed (Delgado 2017b, 2017c) for specific cases of interest: a) the reaching of *GHZ* and *W* states departing from separable states (Delgado 2017c), and b) the enlargement of those states departing from their shorter versions with two less qubits by the integration of an additional separable pair (Delgado 2017b). Nevertheless, those cases have a low complexity due to their reduced number of terms. In the *SU*(2) reduction, despite of simplicity of the operations, still remains the complexity on the large number of blocks operating in the entire state. In addition, the number of different *SU*(2) blocks (not including their *U*(1) phase component) is reduced, the most of them are identical. This fact reflects the underlying complexity of entanglement (Gurvits 2003), particularly when the number of parts grows.

Process to Generate Extended Entanglement in the SU(2) *Reduction Scheme*

In this subsection, we address with the problem to generate (or to reduce, with the inverse operations, due to the reversibility of quantum mechanics), larger entanglement from separable states, as instance. By departing from the 2-separable basic state $|\Psi_0\rangle^{2d}=|\Psi_0,..,\Psi_0\rangle$ (the tensor product of *d* Bell states $|\Psi_0\rangle$), a general procedure (but non unique) to generate the sixteen terms in a general state involving two correspondent pairs (four single qubits) could be stated in the following way. If *i*, *j*, *k* is a permutation of 1, 2, 3, then we apply first a 2-local operation with associated direction *i* on the first correspondent pair *s*, then another on the second correspondent pair *s*' in the direction *j*. After, we apply a 4-local operation on both pairs in the direction *k* (generating entanglement between the two pairs). Finally, we apply a 2-local operation on the first pair *s* in the direction *j*. This procedure gives the sixteen possible terms and certain degree of entanglement:

$$
\begin{aligned}
|\Psi_0\rangle^{2d} \xrightarrow[O_I^{(i,\{s\})}]{H_I^{(i,s)}} & \sum_{t\in\{0,i\}} \alpha_{t,0}^{s} |\Psi_{0,..,t,..,0,..,0}\rangle \xrightarrow[O_I^{(j,\{s'\})}]{H_I^{(j,s')}} \sum_{\substack{t\in\{0,i\}\\ t'\in\{0,j\}}} \alpha_{t,0}^{s}\alpha_{t',0}^{s'} |\Psi_{0,..,t,..,t',..,0}\rangle \\
\xrightarrow[O_{II_a}^{(k,\{s,s'\})}]{H_{II_a}^{(k,s,s')}} & \sum_{\substack{\epsilon,\epsilon'\in C_4\\ t\in\{0,i\}\\ t'\in\{0,j\}}} \alpha_{t,0}^{s}\alpha_{t',0}^{s'}\beta_{\epsilon,\epsilon';t,t'}^{s,s'}\delta_{t,p_{s,k}^{t,\epsilon}}\delta_{t',p_{s',k}^{t',\epsilon'}} |\Psi_{0,..,\epsilon,..,\epsilon',..,0}\rangle \\
\xrightarrow[O_I^{(j,\{s\})}]{H_I^{(j,s)}} & \sum_{\substack{\chi,\epsilon,\epsilon'\in C_4\\ t\in\{0,i\}\\ t'\in\{0,j\}}} \alpha_{t,0}^{s}\alpha_{t',0}^{s'}\beta_{\epsilon,\epsilon';t,t'}^{s,s'}\alpha_{\chi,\epsilon}^{s}\delta_{t,p_{s,k}^{t,\epsilon}}\delta_{t',p_{s',k}^{t',\epsilon'}}\delta_{\epsilon,p_{s,j}^{t,\chi}} |\Psi_{0,..,\chi,..,\epsilon',..,0}\rangle
\end{aligned}
\tag{16}
$$

where C_4={0,1,2,3} and $p^{t,\ \varepsilon}{}_{s,j}$ is the extension of the inverse exchange rule (Delgado, 2017b), but specifying the rule *j* as function of the direction of the interaction involved and the script $t \in \{0, i\}$ is a label specifying each possible inverse. It means, if *j* is the characteristic direction of interaction: $p^{0,i}{}_{s,j}=k=p^{0,k}{}_{s,j}$, $p^{0,0}{}_{s,j}=0=p^{0,j}{}_{s,j}$ and $p^{i,i}{}_{s,j}=i=p^{i,k}{}_{s,j}$, $p^{i,0}{}_{s,j}=j=p^{i,j}{}_{s,j}$. By increasing the number of additional intermediate operations (1-local, 2-local or 4-local), it is possible to extend the coverage on the elements of *SU*(2) for each evolution operator in each subspace. Alternatively, other possible procedures to generate larger entangled states could give more extended entanglement as those cases presented in Delgado (2017b) to generate the *GHZ* and *W* states with genuine entanglement. In any case, the way (clearly the ways) to reach some state by construction departing from a very simple state as $|\Psi_0\rangle^{2d}$ is still open. The last fact suggests how to address a quantum information processing to generate some general states from simpler ones or otherwise, reducing an arbitrary state into another with low entanglement (1 or 2-separable in the current scheme).

But the process will not become sufficiently obvious when the complexity grows together with the size of the system. For an arbitrary state is not always clear how it can be reduced into a separable state, despite from the analysis becomes true it requires both, 1 or 2-local operations, but necessarily, 4-local operations to decompose the possible entangled structure in an arbitrary state. In addition, the advancement on the process is difficult to be continuously monitored because the lack of a universal entanglement measure. We will try to compensate that lack in the next section with the Rényi-Ingarden-Urbanik entropy.

Analysis to Reduce the RIU Entropy through the *SU*(2) Reduction with 1, 2 and 4-Local Operations

In this section we finally analyze how a series of operations based on *SU*(2) reduction are able to transform arbitrary states into simple ones. We use the Rényi-Ingarden-Urbanik entropy to depict the process in terms of the global entropy of the state.

Rényi-Ingarden-Urbanik Entropy

A measure of entanglement for arbitrary number of qubits (*q*-dits in fact) is the RIU entropy (Ingarden and Urbanik 1962) defined as:

$$S_p(|\psi\rangle) = \frac{1}{1-p}\log\left(\sum_{i=1}^{n}|\alpha_i|^{2p}\right), \quad \text{with}: \quad |\psi\rangle = \sum_{i=1}^{n}\alpha_i|\phi_i\rangle \tag{17}$$

We will consider the most common case p=1 as a measure of the entanglement of larger states as those presented in the context of the $SU(2)$ reduction process:

$$S_1(|\psi\rangle) = -\sum_{i=1}^{n} |\alpha_i|^2 \log |\alpha_i|^2, \quad \text{with}: \quad |\psi\rangle = \sum_{i=1}^{n} \alpha_i |\phi_i\rangle \tag{18}$$

The Shannon entropy; clearly these quantities depend on the representation of the states and the basis being used in it, despite they can be minimized through optimal local operations into the minimal RIU entropy (Enríquez, Puchala and Życzkowski 2015). RIU entropy with p=1 for 2d qubits exhibits values ranging from 0 (for separable states) to $\log(2^{2d})$ (for certain maximal entangled states).

Despite this entanglement measure could be useful to provide a dial to sense the entanglement degree, it is not clear how it relates with the inner structures of entanglement. Thus, it will work to have a general and a side-to-side quantification for the entanglement. We are particularly interested on the big picture of the transit from an arbitrary state (commonly entangled) into another finally separable, using only 1, 2, 4-local operations as they could be provided by the $SU(2)$ decomposition scheme.

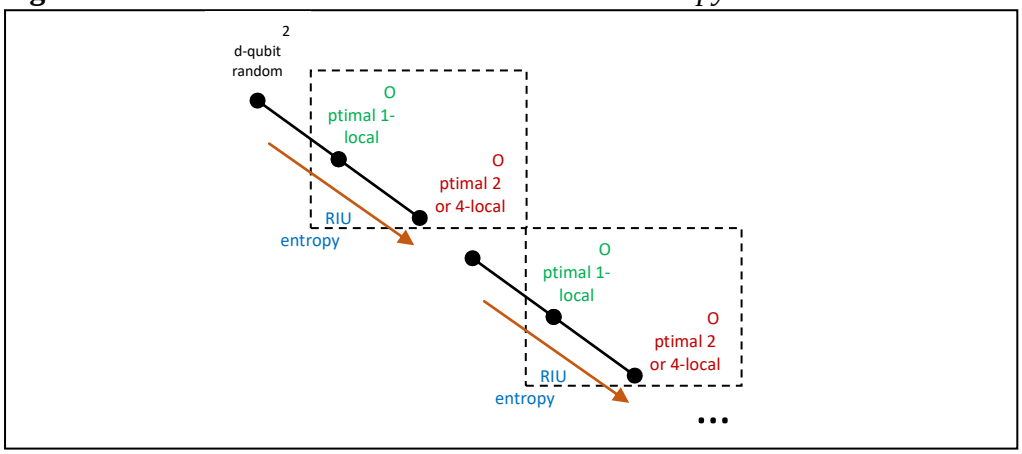
Stochastic Process Reducing RIU Entropy with SU(2) *Reduction Operations*

Following the ideas previously presented, it has been automated a process to reduce the entropy of arbitrary states. It begins by generating a random state in the Haar's measure in 2^{2d-1} dimensions (Diestel and Spalsbury 2014). Then, the state is first processed with 1-local operations to reduce optimally its RIU entropy. After, a stochastic process begins by selecting a 2-local or a 4-local operation, together with a characteristic direction for it. In any case, an optimization problem is solved in order to select the best parameters of such operation in order to minimize its RIU entropy with it. Then, process is always followed with the application of an optimal 1-local operation to reduce again the entropy (2 and 4-local operations includes by extension the 1-local operations in the $SU(2)$ reduction scheme). Clearly, this step process only is able to reduce it in a limited strength because the residuary entanglement involving more complex structures (Gurvits 2003). Thus, the process continues alternating those 2 or 4-local operations followed with a 1-local operation optimally to reduce the RIU entropy (Figure 2).

As was stated, the selection of the local order of operations, the associated direction together with the pairs involved, are completely stochastic in each step. Last process is hard to simulate because the large number of parameters associated to be optimized on each step. Each one lasts out around of two minutes to be numerically processed (for utmost six qubits on an Intel Xeon 3.40 GHz). The process was followed by certain number of operations tracking its RIU entropy. For the six-qubits, we are working with basis of size 64.

Figure 3 depicts such basis elements and their relations under the *SU*(2) reduction combining 2 and 4-local operations, reflecting the complexity in the interactions for this system (only six qubits!). Each script exhibits their composition in terms of Bell states if it is expressed in base 4.

Figure 2. *Stochastic Process to Reduce the RIU Entropy*



Blue arrows match the overall relations settled by any 2-local operation in all possible associated directions and parts (between those basis elements differing in only one digit in the base 4 representation of scripts). Green arrows relate the relations settled by any 4-local operation considering all possible directions and parts (differing in two digits in the scripts).

Figure 4 shows ten simulation experiments with a six-qubits system processed with the last procedure with 300 steps of such optimal operations each one. Note how the RIU entropy reduces rapidly in the first steps. After, the process slows until a casual operation finally reaches a deep reduction of the entropy (resembling a thermodynamic change of state) to then remain almost static until a new remarked reduction. Some of those experiments reach a zero entropy indicating the achievement of a separable state. Clearly, the inverse process can be attained reversing the quantum operations. In any case, we note it is possible to disassemble (or assemble) arbitrary states reaching (or departing from) single separable states.

Figure 3. *Basis Elements for the 2d=6 Case for the SU(2) Reduction and Relations for a) 2-Local Operations (blue), and b) 4-Local Operations (green)*

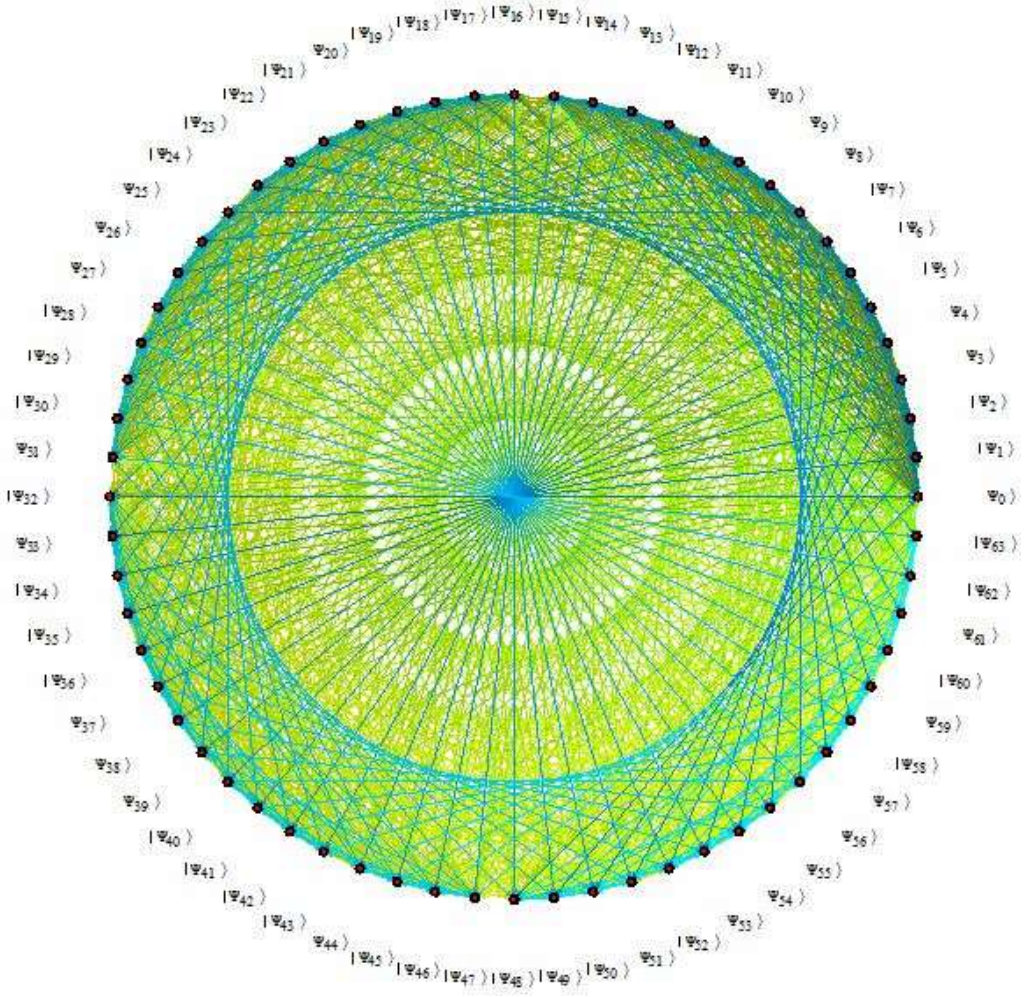
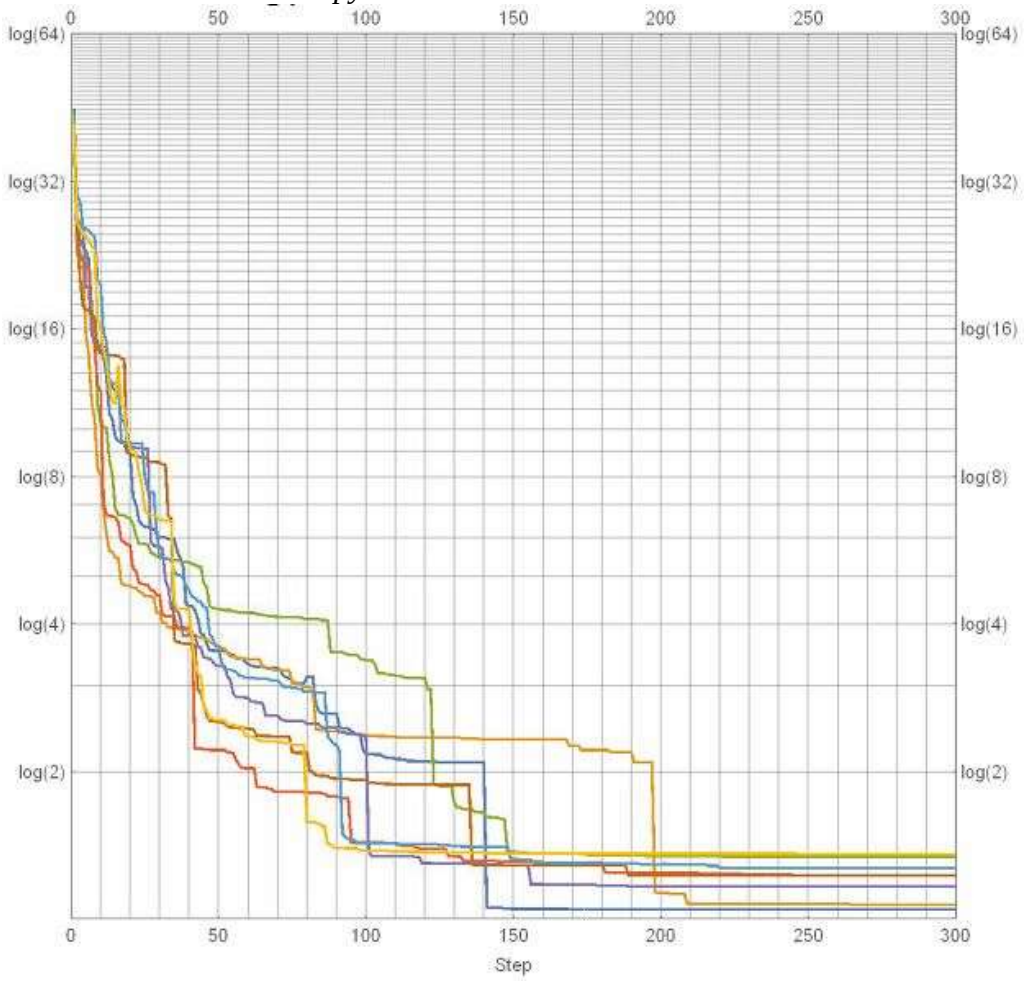


Figure 4. *Ten Random States Evolving through a Series of 300 Steps of the Stochastic Process Combining 2-Local and 4-Local Optimal Operations to Decrease their RIU Entropy*



The process is not direct, to see that it is important understand how the use of information states (more than physical states) works. The evolution by blocks works in parallel on large number of states, instead independently. It means, for the 2-local interactions, there are only two different blocks at the time. In fact, all blocks are of two types, both exchanging one digit in the script of states while other remain unchanged: a) those exchanging 0 and the digit *j* (the associated direction to the interaction; in the crossed interaction the Type III, both coordinates involved defines that direction as a permutation of 1, 2, 3), and b) those exchanging *i*, *j*, the remaining scripts. For the Type $II_{a,b}$ interactions, the situation is similar, but there are eight different blocks related with the exchanges (Delgado 2017a). Despite the complexity, the RIU entropy works as an indicator in the process. The processes depicted in the Figure 4 deeply suggest a certain order in the decreasing of the RIU entropy under this stochastic roadmap. Still more numerical data are precise to reach statistical meaningful in such quantification. Other findings in the last plot are suggested in the deceleration of the RIU entropy reduction around of log(4). Further

evolution departing from this point is erratic in terms of optimality in the reduction of the residual entropy. Six-qubits case is only the beginning. Note other works have attempted similar approaches in the comprehension of the structure of large entangled states by identifying inner structures of partial entanglement (Guhne and Toth 2009, Kraus 2010, Zangi et al. 2018).

Conclusions

The comprehension of entanglement, constitution and measurement are aspects closely related. In addition, limitations associated with the formation of entangled states for large systems suppose a high quest being able to reach or manipulate arbitrary quantum states, because it is closely related with the system and the associated Hamiltonian. Quantum information theory is widely based on this fact, thus the importance to learn about control and processing in many body systems.

In the current work, the use of $SU(2)$ reduction as analysis element to identify and to classify the entangling operations based on the concrete possibilities for the Hamiltonian has becoming valuable to set an automated road to analyze the possibility of processing. In addition, it has been useful to set a common grammar for a group of possible related interactions. Despite the complexity becomes exacerbated by the lack of a general measurement and quantification of entanglement, the use of RIU entropy is still useful to appoint the beginning and the goal of the entire processing, nevertheless it does not give a clear and complete compass in the middle of the road, particularly in terms of optimality. A future work opportunity in this context is a customary factorization for a direct $SU(2^{2d})$ matrix resembling the processing transforming a state into another, which could be uniquely factorized in terms of a matrix basis being formed with the $SU(2)$ blocks present in the reduction formalism.

In the last outcome (Figure 4), it is barely clear there is a notable reduction in the RIU entropy in the first steps of the process by the use of combined 2 and 4-local operations. The decided reduction stops approximately in log(4) to then going on a slower reduction with only casual and sudden entropy fallings. This region appears as the most unclear and complex processing in terms of the RIU entropy reduction with the available operations. Despite, the processing seems going to zero-RIU entropy as a function of time (or the number of steps needed). Still more research and improvements in the computer simulations of this proposed process appears as necessary to get some statistical insights about its qualities, which could be associated in general as the processing of quantum states for large systems. Statistical analysis is suggested in terms of other approaches as invariants (Eltschka et al. 2011), customized factorization (Luo et al. 2014), rank of coefficient matrix (Li and Li 2012), generalized singular value decomposition (De Lathauwer et al. 2000), etc. In any case, still there are many facts to explore in the understanding of entanglement and its

quantification, together with the general dominion of quantum processing, in particular when the size of systems grows.

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