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Cocycles over Non-Autonomous Dynamical Systems in Banach Spaces

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Cocycles over Non-Autonomous Dynamical Systems in Banach Spaces

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Abstract

The aim of this paper is to describe some asymptotic properties for the solutions of evolution equations by means of cocycles over non-autonomous dynamical systems, as generalizations of the skew-evolution cocycles. We present some concepts of instability for cocycles on infinite dimensional spaces. We give characterizations and establish connections between these notions, underlined by examples.

Keywords: Non-autonomous dynamical system, skew-evolution cocycle, ω-growth, ω-decay, uniform exponential instability, exponential instability, (α,β)-instability, (h,k)-instability
Mathematics Subject Classification: 34D05, 93D20

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Introduction

Several concepts of the control theory, as stability, stabilizability, controllability or observability are recently reconsidered, based on the fact that the dynamical systems which describe processes from the real world are more and more complex. Some asymptotic properties that appear in the theory of dynamical systems play an important role in the study of stable and instable manifolds, and, hence, in the study of dichotomy. There are remarkable results due to J.L. Daleckii and M.G. Krein (see [3]), J.L. Massera and J.J. Schaeffer (see [4]) or O. Perron (see [6]).

The skew-evolution semiflows, defined in [5], are generalizations for evolution families and skew-product semiflows. Some researchers, as A.J.G. Bento and C.M. Silva (see [2]), P. Viet Hai (see [13] and [14]) have already adopted the notion and emphasized its applicability. Some results concerning the asymptotic behaviors of skew-evolution semiflows on infinite dimensional spaces were published in [7] and [9]. Various stability properties were studied in [8] and some trichotomy properties in [10]. A study of multivalued non-autonomous dynamical system was done, for example, in [12].

In this paper we consider the case of skew-evolution cocycles over a non-autonomous dynamical system. We define various concepts of instability, such as uniform exponential instability, exponential instability, (α,β)-instability (see [1]), and, a more general concept, the (h,k)-instability. Characterizations and connections between these notions are also given.

Notations and Definitions

Let (X, d) be a complete metric space, $P(X)$ the set of all non-empty subsets of X, V a Banach space and $B(V)$ the space of all V-valued bounded linear operators on V. We denote $Y = X\times V$, id_X the identity map on X, I the identity operator on V and we define the set $T = \{(t,s)\in \mathbf{R}_+^2 \mid t \geq s \geq 0\}$.

Definition 2.1. A map $S : T\times X \to P(X)$ with the properties:

(ds_1) $S(t,t,\cdot) = id_X$, $\forall (t,x)\in \mathbf{R}_+ \times X$;

(ds_2) $S(t,t_0,x) \subseteq S(t,s,S(s,t_0,x))$, $\forall (t,s),(s,t_0)\in T$, $\forall x\in X$

is called *generalized multivalued non-autonomous dynamical system* on X.

Remark 2.2. We will consider the particular case $u : T\times X \to X$ such that:

(ds_1)' $u(t,t,x) = x$, $\forall (t,x)\in \mathbf{R}_+ \times X$;

(ds_2)' $u(t,t_0,x) = u(t,s,s(s,t_0,x))$, $\forall (t,s),(s,t_0)\in T$, $\forall x\in X$,

a mapping that we call *semiflow* associated to the generalized multivalued non-autonomous dynamical system S on X.

Definition 2.3. A mapping $U : T\times X \to B(V)$ which satisfy the conditions:

(c_1) $U(t,t,x) = I$, $\forall (t,x)\in \mathbf{R}_+ \times X$;

(c_2) $U(t,s,u(s,t_0,x))U(s,t_0,x) = U(t,t_0,x)$, $\forall (t,s),(s,t_0)\in T$, $\forall x\in X$

is called *skew-evolution cocycle* over u.

Example 2.4. We consider $\boldsymbol{C} = C(\boldsymbol{R}, \boldsymbol{R})$ the metric space of all continuous functions $f : \boldsymbol{R} \to \boldsymbol{R}$, with the topology of uniform convergence on compact subsets of $\boldsymbol{R}$. Let X be the closure in $\boldsymbol{C}$ of the set $\{x_t \,/\, x_t(s) = x(t+s),\ t,s \in \boldsymbol{R}_+\}$, which is a metric space. We consider the Cauchy problem

$$\begin{cases} v'(t) = A(u_0(t,x))v(t), \ t > 0 \\ v(0) = v_0. \end{cases}$$

where $A : X \to B(V)$ is an operator, $v_0 \in DomA$ and $u_0 : \boldsymbol{R}_+ \times X \to P(X)$ is defined by $u_0(t,x) = x_t$. We consider a C_0-semigroup H defined by $H(t)v = \sum_{n=0}^{\infty} e^{-n^2\pi^2 t} \langle v, e_n \rangle e_n$, where $\{e_n\}_{n \in \boldsymbol{N}}$, $e_0 = 1$ and $e_n(y) = \sqrt{2} \cos 2ny$, $y \in (0,1)$, is the orthonormal basis of the separable Hilbert space $V = L^2(0,1)$. Let us define $U_0 : T \times X \to B(V)$, by $U_0(t,x)v = H\left(\int_0^t x(s)ds\right)v$. For all v_0, we have that $v(t) = U_0(t, x)v_0$, $t>0$, is a strong solution of the Cauchy problem. We define $u(t,s,x) = u_0(t-s, x)$ and $U(t,s,x) = U_0(t-s, x)$. We have that U is a skew-evolution cocycle over u.

Definition 2.5. A skew-evolution cocycle U has ω*-decay* if there exists a non decreasing function $\omega : \boldsymbol{R}_+ \to \boldsymbol{R}_+^*$ with $\lim_{t \to \infty} \omega(t) = \infty$ such that:

$\|v\| \le \omega(t - t_0)\|U(t,t_0,x)v\|$, $\forall (t,t_0) \in T$, $\forall (x,v) \in Y$

Remark 2.6. If U has ω-decay, then the $-\lambda$-shifted skew-evolution cocycle denoted $U_{-\lambda}$, $\lambda < 0$, and defined by $U_{-\lambda}(t,s,x) = e^{\lambda(t-s)}U(t,s,x)$ has also ω-decay.

Remark 2.7. The asymptotic property given by Definition 2.5 is equivalent with the property of exponential decay defined in [11].

Various Types of Instability

Let U be a skew-evolution cocycle over u. We define the set E of all mappings $f : \boldsymbol{R}_+ \to \boldsymbol{R}_+^*$ for which there exists $\alpha \in \boldsymbol{R}_+$ such that $f(t) = e^{\alpha t}$.

Definition 3.1. U is said to be *uniformly exponentially instable* if there exist some constants $N \ge 1$ and $\nu > 0$ such that:

$$e^{\nu(t-t_0)}\|v\| \le N\|U(t,t_0,x)v\|, \ \forall (t,t_0) \in T, \ \forall (x,v) \in Y.$$

Definition 3.2. U is *exponentially instable* if there exist a mapping $N : \boldsymbol{R}_+ \to [1,\infty)$ and a constant $\nu > 0$ such that:

$$e^{\nu(t-t_0)}\|v\| \le N(t)\|U(t,t_0,x)v\|, \ \forall (t,t_0) \in T, \ \forall (x,v) \in Y.$$

Definition 3.3. U is *(α,β)-exponentially instable* if there exist some constants $N \geq 1$ and $\alpha,\beta > 0$ such that:

$$\|v\| \leq Ne^{-\alpha t}e^{\beta t_0}\|U(t,t_0,x)v\|,\ \forall(t,t_0)\in T,\ \forall(x,v)\in Y.$$

Definition 3.4. U is *(h,k)-instable* if there exist a constant $N \geq 1$ and two continuous mappings $h,k : \boldsymbol{R}_+ \to \boldsymbol{R}_+^*$ such that:

$$h(t-t_0)\|v\| \leq Nk(t)\|U(t,t_0,x)v\|,\ \forall(t,t_0)\in T,\ \forall(x,v)\in Y.$$

Some connections between the previous notions are given by

Remark 3.5. If we consider $h \in E$ and $\alpha = 0$, then the skew-evolution cocycle U is uniformly exponentially instable.

Remark 3.6. If we consider $h \in E$, then the skew-evolution cocycle U is exponentially instable.

Remark 3.7. If we take $h,k \in E$, then skew-evolution cocycle U is (α,β)-exponentially instable.

Hence, it follows that the notion of (h,k)-instability generalizes the concepts of uniform exponential instability, exponential instability and (α,β)-exponential instability.

Other connections are given by the following statements.

Remark 3.8. The property of uniform exponential instability of a skew-evolution cocycle implies the (α,β)-exponential instability, which implies further the property of exponential instability.

The converse statement is not always true, as shown in the next example.

Example 3.9. Let $f : \boldsymbol{R}_+ \to \boldsymbol{R}_+^*$ be a decreasing function with the property that there exists $\lim\limits_{t\to\infty} f(t) = l > 0$. Let $\mu > f(0)$. We will consider the metric space X defined in Example 2.4 and the semiflow $u : T \times X \to X$ defined by $u(t,s,x)(\tau) = x_{t-s}(\tau)$. Let $V = \boldsymbol{R}$. The mapping $U : T \times X \to B(\boldsymbol{R})$, defined by $U(t,s,x)v = \dfrac{e^{3t-2t\cos t}}{e^{3s-2s\cos s}}e^{\int_s^t x(\tau-s)d\tau}v$, is a skew-evolution cocycle over u. As

$$\|U(t,t_0,x)v\| = e^{3t-3t_0-2t\cos t+2t_0\cos t_0+\int_{t_0}^t x(\tau-t_0)d\tau}|v| \geq e^{t-t_0}e^{l(t-t_0)}|v| = e^{(1+l)t}e^{-(1+l)s}|v|$$

hold for all $(t,t_0)\in T$ and all $(x,v)\in Y$, it follows that U is (α,β)-exponentially instable with $N = 1$ and $\alpha = \beta = 1+l$. On the other hand, if we suppose that U is uniformly exponentially instable, we obtain, according to **Definition 3.1,** that there exist $N \geq 1$ and $\nu > 0$ such that relation

$$Ne^{3t-3t_0-2t\cos t+2t_0\cos t_0+\int_{t_0}^t x(\tau-t_0)d\tau}|v| \geq e^{\nu(t-t_0)}|v|$$

holds $\forall(t,t_0)\in T$, $\forall(x,v)\in Y$. If $t = 2n\pi$ and $t_0 = 2n\pi-\pi$, we obtain

$$Ne^{-4n\pi+3\pi} \geq e^{\nu\pi}e^{\int_{2n\pi-\pi}^{2n\pi} x(\tau-2n\pi+\pi)d\tau} \geq e^{(\nu-\mu)\pi}$$

which, for $n \to \infty$ leads to a contradiction. Hence, U is not uniformly exponentially instable.

In order to introduce the following instability concepts, let us consider that U is *strongly measurable*, which means that, for every $(t_0, x, v) \in \boldsymbol{R}_+ \times Y$, the mapping $s \mapsto \|U(s, t_0, x)\|$ is measurable on $[t_0, \infty)$.

Definition 3.10. U is (h,k)*-integrally instable* if there exist a constant $D \geq 1$ and two continuous mappings $h,k : \boldsymbol{R}_+ \to \boldsymbol{R}_+^*$, where h *is* a non decreasing function which satisfies the property $h(t+\tau) \leq h(t)h(\tau)$, $\forall t, \tau \in \boldsymbol{R}_+$, such that:

$$\int_{t_0}^{t} h(t-\tau)\|U(\tau, t_0, x)v\| d\tau \leq Dk(t)\|U(t, t_0, x)v\|, \ \forall (t, t_0) \in T, \ \forall (x, v) \in Y.$$

A particular case is given by

Definition 3.11. U is said to be *integrally instable* if there exists a mapping $M : \boldsymbol{R}_+ \to \boldsymbol{R}_+^*$ such that:

$$\int_{0}^{t} \|U(\tau, t_0, x)v\| d\tau \leq M(t)\|U(t, t_0, x)v\|, \ \forall (t, t_0) \in T, \ \forall (x, v) \in Y$$

Main Results

An integral characterization for the notion of exponential instability is given by means of the shifted skew-evolution cocycle in

Theorem 4.1. *A strongly measurable skew-evolution cocycle U with ω-decay is exponentially instable if and only if there exists a constant $\lambda > 0$ such that the $-\lambda$-shifted skew-evolution cocycle $U_{-\lambda}$ is integrally instable.*

Proof. Necessity. Let us define $\lambda = -\dfrac{\nu}{2} < 0$, where the existence of the constant ν is assured by Definition 3.2. We obtain

$$\int_{0}^{t} \|U_{-\lambda}(\tau, t_0, x)v\| d\tau = \int_{0}^{t} e^{\lambda(\tau - t_0)}\|U(\tau, t_0, x)v\| d\tau \leq$$

$$\leq N(t)\int_{t_0}^{t} e^{\lambda(\tau - t_0)} e^{-\nu(t-\tau)}\|U(t, t_0, x)v\| d\tau =$$

$$= N(t)\int_{t_0}^{t} e^{\lambda(t - t_0)} e^{-\lambda(t-\tau)} e^{-\nu(t-\tau)}\|U(t, t_0, x)v\| d\tau =$$

$$= N(t)\|U_{-\lambda}(t, t_0, x)v\|\int_{t_0}^{t} e^{\lambda(t-\tau)} d\tau \leq -\frac{N(t)}{\lambda}\|U_{-\lambda}(t, t_0, x)v\|$$

for all $(t, t_0) \in T$ and all $(x, v) \in Y$. Hence, the shifted skew-evolution cocycle is integrally instable.

Sufficiency. Let us denote $K=\int_0^1 e^{\lambda u}\omega(u)du$, where function ω is given by **Definition 2.5,** as, according to the hypothesis, U has ω-decay. We obtain successively

$$K\|v\|=\int_{t_0}^{t_0+1} e^{\lambda(\tau-t_0)}\omega(\tau-t_0)\|U(t_0,t_0,x)v\|d\tau\leq\int_{t_0}^{t_0+1} e^{\lambda(\tau-t_0)}\|U(\tau,t_0,x)v\|d\tau\leq$$

$$\leq M(t)\|U_{-\lambda}(t,t_0,x)v\|=M(t)e^{\lambda(t-t_0)}\|U(t,t_0,x)v\|$$

for all $(t,t_0)\in T$ and all $(x,v)\in Y$, function M being given by Definition 3.11. It follows that U is exponentially instable, which ends the proof.

Corollary 4.2. *In the hypothesis of Theorem 4.1, we have:*

(i) if $h,k\in E$ *and are given by* $t\to e^{\alpha t}$, *respectively* $t\to Me^{\alpha t}$ *with* $M\geq 1$, *the skew-evolution cocycle U is uniformly exponentially instable;*

(ii) if $h\in E$, *the skew-evolution cocycle U is exponentially instable;*

(iii) if $h,k\in E$ *and are given by* $t\to e^{\alpha t}$, *respectively* $t\to Me^{\beta t}$ *with* $M\geq 1$ *and* $\beta>\alpha$, *the skew-evolution cocycle U is* (α,β)*-exponentially instable.*

The next result establishes a relation between the properties of (h,k)-instability and (h,k)-integral instability.

Theorem 4.2. *A* (h,k)*-integrally instable skew-evolution cocycle U with* ω*-decay is* (h,k)*-instable.*

Proof. Let us consider that U has ω-decay, which, according to Definition 2.5 assures the existence of function ω. As U is (h,k)-integrally instable, following relations hold

$$Dh(t-s)\|U(s,t_0,x)v\|=\int_0^1 h(t-s)\omega(\tau)\|U(s,t_0,x)v\|d\tau\leq$$

$$\leq\int_0^1 h(t-\tau)h(\tau-s)\omega(\tau)\|U(s,t_0,x)v\|d\tau\leq h(t)\int_0^1 h(\tau)\omega(\tau)\|U(s,t_0,x)v\|d\tau=$$

$$=\int_s^{s+1} h(u-t_0)\omega(u-s)\|U(s,t_0,x)v\|du\leq\int_0^t h(u-s)\|U(u,t_0,x)v\|du\leq$$

$$\leq Dk(t)\|U(t,t_0,x)v\|,$$

for all $t\geq s+1>s\geq 0$ and all $(x,v)\in Y$. On the other hand, for $t\in[s,s+1)$

$\|U(t,t_0,x)v\|\geq\omega(t-s)\|U(s,t_0,x)v\|\geq\omega(1)\|U(s,t_0,x)v\|$, $\forall(x,v)\in Y$.

Hence, it follows that U is (h,k)-instable, which ends the proof.

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