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Abstract

In the Introduction, we recall some earlier, as well as recent results that represent the background of the present work. Then, applications of Mazur-Orlicz theorem are presented. Two domain-spaces are involved: spaces of analytic functions, and L^1 spaces. Results on the existence of the solutions of some Markov moment problem are stated and respectively proved. On the other hand, one uses approximation theorems on unbounded intervals in order to point out results on positive polynomials of one real variable to the case of several real dimensions. Approximation by sums of tensor products of positive polynomials in each separate variable is applied. The last result is a generalization of one of our earlier results, from one to several variables. The domain-space is a space of analytic functions in a polydisc, and the target one is a space of selfadjoint operators. The structure of all the spaces involved is of a real ordered space. One of the goals of these applications is to point out the relationship between the Markov moment problem and Mazur-Orlicz principle. These two results seem to be quite different.

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Introduction and known results

Applying extension Hahn-Banach type results in existence questions concerning the moment problem is a well-known technique: [1], [2], [5], [9]-[14], [17]-[21]. One of the most useful results is Lemma of the majorizing subspace (see [6] Section 5.1.2 for the proof of the lattice-version of this lemma; see also [23]). It says that *if* f *is a linear positive operator on a subspace* S *of the ordered vector space* X, *the target space being an order complete vector lattice* Y, *and for each* $x \in X$ *there is* $s \in S, x \le s$, *then* f *has a linear positive extension* $F : X \to Y$. Another geometric remark is that in the real case, the sublinear functional from Hahn-Banach theorem can be replaced by a convex one. The theorem remains valid when the convex dominating functional is defined on a convex subset A with some qualities with respect to the subspace S (for instance: $S \cap ri(A) \ne \Phi$), where $ri(A)$ is the relative interior of A. Here we recall an answer published in 1978 ([15]), without loosing convexity, but strongly generalizing the classical result. The first detailed proof was published in 1983 [16]. The proof of a similar result, in terms of the moment problem was published in [18]. Here we recall the general statement from [15]. One of the reasons is that many other results are consequences of this theorem, including Bauer's theorem [23], Namiokas's theorem and abstract moment problem-results published firstly in [17]. Part of these generalizations of the Hahn-Banach principle appears in the present work too. For uniqueness of the solution of moment problems see [3], [5], [7], [8], [24]. Fixed point theorems and iterative methods in moment problem are used in [4]. For constructing the solutions see [12] and [21]. Other main results are contained in [22]. Throughout this first part, X will be a real vector space, Y an order-complete vector lattice, $A, B \subset X$ convex subsets, $q : A \to Y$ a concave operator, $p : B \to Y$ a convex operator, $S \subset X$ a vector subspace, $f : S \to Y$ a linear operator.

Theorem 1.1. *Assume that:* $f|_{S \cap A} \ge q|_{S \cap A}$, $f|_{S \cap B} \le p|_{S \cap B}$.

The following assertions are equivalent:

(a) *there is a linear extension* $F : X \to Y$ *of the operator* f *such that:* $F|_A \ge q, \quad F|_B \le p$;

(b) *there are* $p_1 : A \to Y$ *convex and* $q_1 : B \to Y$ *concave operators such that for all*

$$(\rho, t, \lambda', a_1, a', b_1, b', v) \in [0,1]^2 \times (0, \infty) \times A^2 \times B^2 \times S,$$

we have:

$$(1-t)a_1 - t b_1 = v + \lambda'[(1-\rho)a' - \rho b'] \Rightarrow$$
$$(1-t)p_1(a_1) - t q_1(b_1) \ge f(v) + \lambda'[(1-\rho)q(a') - \rho\, p\,(b')].$$

The minus-sign appears to construct a convex operator in the left-hand side member and a concave operator in the right side. The idea of sandwich theorem on arbitrary convex subsets A, B is clear. Most of the applications hold for linear positive operators on linear ordered spaces (X, X_+), when we take $A = X_+, q \equiv 0, B = X, p$ a suitable convex operator (a vector-valued norm, a sublinear operator), which "measures the continuity" of the extension F. One obtains the following result related to the theorem of H. Bauer ([23], Section 5.4).

Theorem 1.2. *Let X be a preordered vector space with its positive cone X_+, $p : X \to Y$ a convex operator, $S \subset X$ a vector subspace, $f : S \to Y$ a linear positive operator. The following assertions are equivalent:*

(a) *there is a linear positive extension $F : X \to Y$ of f such that $F(x) \le p(x)\, \forall x \in X$;*

(b) $f(x') \le p(x)$ *for all $(x', x) \in S \times X$ such that $x' \le x$.*

Now we can state the main results on the abstract moment problem [17].

Theorem 1.3. *Let $X, Y, P : X \to Y$ be as in Theorem 1.2, $\{x_j\}_{j \in J} \subset X, \{y_j\}_{j \in J} \subset Y$ given families. The following assertions are equivalent:*

(a) *there is a linear positive operator $F : X \to Y$ such that $F(x_j) = y_j\ \forall j \in J, F(x) \le P(x)\, \forall x \in X$;*

(b) *for any finite subset $J_0 \subset J$ and any $\{\lambda_j\}_{j \in J_0} \subset R$, we have:*

$$\sum_{j \in J_0} \lambda_j x_j \le x \Rightarrow \sum_{j \in J_0} \lambda_j y_j \le P(x)$$

A clearer sandwich-moment problem variant is the following one.

Theorem 1.4. *Let $X, Y, \{x_j\}_{j \in J}, \{y_j\}_{j \in J}$ be as in Theorem 1.3 and $F_1, F_2 \in L(X, Y)$ two linear operators. The following statements are equivalent:*

(a) *there is a linear operator $F \in L(X, Y)$ such that $F_1(x) \le F(x) \le F_2(x)\, \forall x \in X_+, F(x_j) = y_j\ \forall j \in J$;*

(b) *for any finite subset $J_0 \subset J$ and any $\{\lambda_j\}_{j \in J_0} \subset R$, we have:*

$$\left(\sum_{j \in J_0} \lambda_j x_j = \varphi_2 - \varphi_1, \varphi_1, \varphi_2 \in X_+ \right) \Rightarrow \sum_{j \in J_0} \lambda_j y_j \le F_2(\varphi_2) - F_1(\varphi_1).$$

Now we state the following generalization of Mazur-Orlicz theorem [17].

Theorem 1.5. *Let $X, Y, \{x_j\}_{j \in J}, \{y_j\}_{j \in J}$ be as in Theorem 1.4, $P : X \to Y$ a sublinear operator. The following statements are equivalent:*

(a) *there is a linear positive operator $F \in L(X, Y)$ such that $F(x_j) \ge y_j, j \in J, F(x) \le P(x), x \in X$;*

(b) *for any finite subset $J_0 \subset J$ and any $\{\lambda_j\}_{j \in J_0} \subset R_+$, we have*

$$\sum_{j\in J_0}\lambda_j x_j \le x \Rightarrow \sum_{j\in J_0}\lambda_j y_j \le P(x).$$

We recall the following polynomial approximation results on unbounded subsets.

Theorem 1.6. (see [13] and [14]-Lemma 1.3 (d)) *If* $\varphi \in C_0([0,\infty)\times[0,\infty))$ *is a nonnegative continuous function with compact support, then there exists a sequence* $(p_m)_m$ *of positive polynomials on* $[0,\infty)\times[0,\infty)$, *such that*

$p_m(t) > \varphi(t)\, \forall t \ge 0, \forall m \in Z_+,\ p_m \to \varphi$

uniformly on compact subsets of $[0,\infty)\times[0,\infty)$.

The idea of the proof is to add the ∞ point and to apply the Stone-Weierstrass Theorem to the subalgebra generated by the functions $\exp(-mt_1 - nt_2),\ m,n \in Z_+$. Then one uses for each such $\exp$ -function suitable majorizing or minorizing partial sums-polynomials.

We recall that the corresponding statements for uniform approximation on compact subsets of $[0,\infty)$ of functions $\psi \in C_0([0,\infty))$ holds (see [13], [14] for the proof and details). The results of the present work are using the statements mentioned above.

Theorem 1.7. *Let* $A \subset R^n$ *be a closed subset and* ν *a positive regular Borel determinate measure on* A, *with finite moments of all orders. Then for any* $\psi \in (C_0(A))_+$, *there is a sequence* $(p_m)_m$ *of polynomials on* A, $p_m \ge \psi$, $p_m \to \psi$ *in* $L^1_\nu(A)$. *We have*

$$\lim \int_A p_m d\nu = \int_A \psi\, d\nu,$$

the cone P_+ *of positive polynomials is dense in* $(L^1_\nu(A))_+$ *and* P *is dense in* $L^1_\nu(A)$.

Theorem 1.8. *Let* $\nu = \nu_1 \times \nu_2$ *be a product of two determinate positive regular Borel measures on* R, *with finite moments of all natural orders. Then any positive continuous function with compact support is approximated in* $L^1_\nu(R^2)$ *by means of sums of tensor products* $p_1 \otimes p_2$, p_j *positive polynomial on the real line, in variable* t_j, $j = 1,2$.

Remark. Theorem 1.8 remains valid when we replace R^2 by R^2_+. Moreover, in the latter case the convergence is uniform on the compact support of the function.

Applications of Mazur-Orlicz theorem

Let X be the space of all analytic functions

$$\varphi(z)=\sum_{n\in N}\alpha_n z^n,\ \alpha_n \in R, |z|<b,$$

the functions φ being continuous in the closed disk $\{|z|\le b\}$. The positive cone X_+ consists of all power series with nonnegative coefficients. On the other hand, let H be a complex Hilbert space, $B\in \mathrm{A}(H)$ a selfadjoint operator,

$$Y_1=\{U\in \mathrm{A}(H); UB=BU\}, Y=\{U\in Y_1; UV=VU, \forall V\in Y_1\}. \qquad (1)$$

It is known ([6], [10]) that Y is a commutative algebra and an order complete vector lattice with respect to the restriction to Y of the usual order relation on the real vector space of all selfadjoint operators. Denote $\varphi_j(z)=z^j, |z|\le b,\ j\in \mathrm{N}$. Let $(U_j)_{j\in \mathrm{N}}$ be a sequence in Y_+.

Theorem 2.1 *Let* $A\in Y_+, \|A\|<b, b>1, \varepsilon>0.$ *Consider the following statements:*

(a) *there exists a linear positive operator* $F\in L_+(X,Y)$ *such that*

$$F(\varphi_j)\ge U_j,\ j\in \mathrm{N}, F(\psi)\le b(bI-A)^{-1}\|\psi\|_\infty+\varepsilon\psi(I), \forall \psi\in X;$$

(b) *the following inequalities hold:*

$$U_j\le A^j+\varepsilon\cdot I,\ j\in \mathrm{N};$$

(c) there exists a linear positive operator $F\in L_+(X,Y)$ such that

$$F(\varphi_j)\ge U_j,\ j\in \mathrm{N}, F(\psi)\le \sum_{n\in \mathrm{N}}|\gamma_n|\cdot A^n+\varepsilon|\psi|(I), \psi=\sum_{n\in \mathrm{N}}\gamma_n\varphi_n\in X.$$

Then $(c)\Leftrightarrow(b)\Rightarrow(a)$

Proof. To prove that $(b)\Rightarrow(a)$, we apply $(b)\Rightarrow(a)$ of Theorem 1.5. Verifying the conditions (b) of the latter theorem, and using Cauchy's inequalities, one deduces:

$$\sum_{j\in J_0}\lambda_j\varphi_j \le \psi = \sum_{n\in N}\gamma_n\varphi_n,\ \lambda_j \ge 0, U_j \le A^j + \varepsilon I \Rightarrow \gamma_n \ge 0 \quad \forall n \in N,$$

$$\lambda_j \le \gamma_j \ \le \frac{\|\psi\|}{b^j}, j \in J_0 \Rightarrow \sum_{j\in J_0}\lambda_j U_j \le \|\psi\|\cdot\left(\sum_{j\in J_0}\frac{A^j}{b^j}\right) + \varepsilon\left(\sum_{j\in J_0}\gamma_j\right)\cdot I \le$$

$$\left(\sum_{n\in N}\frac{A^n}{b^n}\right)\cdot\|\psi\| + \varepsilon\psi(1)\cdot I = \left(I - \frac{A}{b}\right)^{-1}\|\psi\| + \varepsilon\psi(1)I =$$

$$b(bI-A)^{-1}\|\psi\| + \varepsilon\psi(1)I = b(bI-A)^{-1}\|\psi\| + \varepsilon\psi(I) = P(\psi), \psi \in X.$$

Now the first conclusion follows via Theorem 1.5. On the other hand, the implication $(c) \Rightarrow (b)$ is almost obvious, since

$U^j \le F(\varphi_j) \le A^j + \varepsilon\varphi_j(I) = A^j + \varepsilon I,\ j \in N.$

It remains to prove the converse, that is $(b) \Rightarrow (c)$. To this end, we apply $(b) \Rightarrow (a)$ of Theorem 1.5 once more. We have the following implications:

$$\sum_{j\in J_0}\lambda_j\varphi_j \prec \psi = \sum_{n\in N}\gamma_n\cdot\varphi_n,\ \lambda_j \in R_+,\ j \in J_0 \Rightarrow$$

$$\lambda_j U_j \le \lambda_j A^j + \varepsilon\lambda_j I \le \gamma_j \cdot A^j + \varepsilon\cdot\gamma_j\cdot I,\ j \in J_0 \Rightarrow$$

$$\sum_{j\in J_0}\lambda_j U_j \le \sum_{n\in N}\gamma_n A^n + \varepsilon\left(\sum_{n\in N}\gamma_n\right)\cdot I \le$$

$$\sum_{n\in N}|\gamma_n|A^n + \varepsilon|\psi|(I) = |\psi|(A) + \varepsilon|\psi|(I) =: P(\psi).$$

Application of Theorem 1.5 yields the existence of a linear operator F with the properties mentioned at point (c). This concludes the proof. □

Using similar arguments, one proves the following variant of Theorem 2.1 in several dimensions (see also [12]). Let X be the space of all absolutely convergent power series in the polydisc $\prod_{k=1}^{n}\{|z_k| < \rho_k\}$, with real coefficients, continuous up to the boundary, $A_1,\ldots,A_n$ positive commuting selfadjoint operators, such that

$\|A_k\| < \rho_k,\ \rho_k > 1,\ k = 1,\ldots,n.$

One denotes:

$$Y_1 = \{U \in \mathcal{A}(H); UA_k = A_kU,\ k = 1,\ldots,n\},$$
$$Y = \{U \in Y_1; UV = VU,\ \forall V \in Y_1\}. \qquad (2)$$

Let $\varphi_j(z_1,\ldots,z_n) = z_1^{j_1}\cdots z_n^{j_n},\ j = (j_1,\ldots,j_n) \in N^n,\ \{B_j\}_{j\in N^n} \subset Y_+.$

Theorem 2.2 *Consider the following statements:*

(a) *there exists a linear positive operator* $F \in L_+(X,Y)$ *such that*

$$F(\varphi_j) \geq B_j \ \forall j \in \mathrm{N}^n, \ F(\psi) \leq \|\psi\| \cdot \prod_{k=1}^{n} \rho_k (\rho_k I - A_k)^{-1}, \quad \forall \psi \in X;$$

(b) *the following inequalities hold*

$$B_j \leq A_1^{j_1} \cdots A_n^{j_n} + \varepsilon \cdot I, \forall j = (j_1, ..., j_n) \in \mathrm{N}^n;$$

(c) *there is a linear operator* $F \in L(X,Y)$ *such that*

$$F(\varphi_j) \geq B_j, \ j \in \mathrm{N}^n, \ F(\psi) \leq |\psi|(A_1, ..., A_n) + \varepsilon |\psi|(I, ..., I), \forall \psi \in X.$$

Then $(c) \Leftrightarrow (b) \Rightarrow (a)$.

We consider an application to the space $X = L_\mu^1(M), \ Y = R, \mu$ being a $\sigma -$ finite positive measure on the measurable space M. The result is valid for not necessary positive functions $\varphi_j, \ j \in J.$ Note that all such statements do not involve polynomials.

Theorem 2.3. *Let*

$$X = L_\mu^1(M), \ \mu \geq 0, \ \{\varphi_j\}_{j \in J} \subset X, \ \{y_j\}_{j \in J} \subset R, \ \mu$$

being a $\sigma -$ *finite measure. Assume that the intersection of the supports of two different functions* $\varphi_{j_k} \neq \varphi_{j_l}$ *have measure zero. The following statements are equivalent:*

(a) *there exists* $h \in L_\mu^\infty(M)$ *such that*

$$0 \leq h(x) \leq 1 \ \mu - a.e., \int_M h \varphi_j d\mu \geq y_j, \ j \in J;$$

(b) *the following inequalities hold*

$$y_j \leq \int_M \varphi_j^+ d\mu, \ j \in J.$$

Proof. The implication $(a) \Rightarrow (b)$ is almost obvious, because of the qualities of h. For the converse, let $J_0 \subset J$ be a finite subset and $\{\lambda_j; \ j \in J_0\} \subset R_+$ such that $\sum_{j \in J_0} \lambda_j \varphi_j \leq \psi.$ Using the hypothesis on the supports, we deduce

$$\sum_{j\in J_0}\lambda_j\varphi_j^+ = \left(\sum_{j\in J_0}\lambda_j\varphi_j\right)^+ \le \psi^+ \Rightarrow$$

$$\sum_{j\in J_0}\lambda_j y_j \le \sum_{j\in J_0}\lambda_j \int_M \varphi_j^+ d\mu = \int_M \left(\sum_{j\in J_0}\lambda_j\varphi_j\right)^+ d\mu \le$$
$$\le \int_M \psi^+ d\mu \le \int_M |\psi| d\mu = P(\psi).$$

We have used the fact that the scalars $\lambda_j,\ j\in J_0$ are nonnegative. Applications of Theorem 1.3 to $P(\psi)=\|\psi\|_1$ leads to the existence of a linear positive form F of norm at most one, such that $F(\varphi_j)\ge y_j,\ j\in J.$ This functional has a representation by means of a function h with the qualities stated at point (a). This concludes the proof. □

Markov Moment Problems on Concrete Spaces and Approximation

The first result of this section is a variant of Theorem 6 [20], having a similar proof. One dimensional variant of lemma 1.6 stated above, as well as Stone Weierstrass and Luzin's theorems are used along the proof. The form of positive polynomials on $[0,\infty)$ [1] is also applied. This make possible to prove the following result, similar to that from the one-dimensional case, although in several dimensions, positive polynomials on $R_+^n, n\ge 2,$ have not a simple representation involving sums of squares. The idea is to approximate positive continuous functions with compact support by sums of tensor products of positive polynomials on $[0,\infty),$ in each separate variable. For details, see [1], [3], [14], [20]. An improved proof will be published soon. Notice that this method works for measures with unbounded support too (see Theorem 3.2).

Theorem 3.1 *Let* $K_1\subset R_+, K_2\subset R_+$ *be compact subsets and* $K=K_1\times K_2.$ *Let* $(y_{j,k})_{(j,k)\in \mathrm{N}^2}$ *be a sequence of real numbers. Let* ν *be the product of two regular positive Borel* $\sigma-$ *finite* M *determinate measures on* $[0,\infty),$ *with finite moments of all orders.*

The following statements are equivalent:

(a) *there exists*

$$h\in L^\infty_{\nu|K}(K),\ 0\le h(t_1,t_2)\le 1,\ \iint_K t_1^j t_2^k h(t_1,t_2)d\nu = y_{j,k},\ (j,k)\in \mathrm{N}^2;$$

(b) *for any finite subsets* $J_1,J_2\subset \mathrm{N},$ *and any* $\{\alpha_j\}_{j\in J_1}, \{\beta_k\}_{k\in J_2},$ *we have:*

$$0 \le \sum_{\substack{i,j \in J_1, \\ k,l \in J_2}} \alpha_i \alpha_j \beta_k \beta_l y_{(i+j,k+l)} \le \sum_{\substack{i,j \in J_1, \\ k,l \in J_2}} \alpha_i \alpha_j \beta_k \beta_l \iint_K t_1^{i+j} t_2^{k+l} d\nu;$$

$$0 \le \sum_{\substack{i,j \in J_1, \\ k,l \in J_2}} \alpha_i \alpha_j \beta_k \beta_l y_{(i+j+1,k+l)} \le \sum_{\substack{i,j \in J_1, \\ k,l \in J_2}} \alpha_i \alpha_j \beta_k \beta_l \iint_K t_1^{i+j+1} t_2^{k+l} d\nu;$$

$$0 \le \sum_{\substack{i,j \in J_1, \\ k,l \in J_2}} \alpha_i \alpha_j \beta_k \beta_l y_{(i+j,k+l+1)} \le \sum_{\substack{i,j \in J_1, \\ k,l \in J_2}} \alpha_i \alpha_j \beta_k \beta_l \iint_K t_1^{i+j} t_2^{k+l+1} d\nu;$$

$$0 \le \sum_{\substack{i,j \in J_1, \\ k,l \in J_2}} \alpha_i \alpha_j \beta_k \beta_l y_{(i+j+1,k+l+1)} \le \sum_{\substack{i,j \in J_1, \\ k,l \in J_2}} \alpha_i \alpha_j \beta_k \beta_l \iint_K t_1^{i+j+1} t_2^{k+l+1} d\nu$$

Using approximation results stated in the end of Section 1, one can prove the following general result involving L^1- norm. Let $\nu = \nu_1 \times \nu_2,$ where $\nu_j,\ j = 1,2$ are positive Borel regular $M-$ determinate measures on $R,$ with finite moments of all natural orders. Let

$$\varphi_{j,k}(t_1,t_2) = t_1^j t_2^k,\ (j,k) \in \mathrm{N}^2,\ (t_1,t_2) \in R^2.$$

Let Y be an order complete Banach lattice with solid norm, and $(y_{j,k})_{(j,k)\in \mathrm{N}^2}$ a sequence in $Y.$ Assume that the convergence with respect to the order relation implies the convergence in the topology of $Y,$ for sequences in $Y.$

Theorem 3.2. *Let $F_2 : L^1_\nu(R^2) \to Y$ be a positive linear bounded operator. The following statements are equivalent:*

(a) *there exists a unique linear operator $F : L^1_\nu(R^2) \to Y,$ such that*

$$F(\varphi_{j,k}) = y_{j,k},\ \forall (j,k) \in \mathrm{N}^2,\ 0 \le F(\psi) \le F_2(\psi),\ \psi \in \left(L^1_\nu(R^2)\right)_+,\ \|F\| \le \|F_2\|;$$

(b) *for any finite subsets $J_1, J_2 \subset \mathrm{N},$ and any $\{\alpha_j\}_{j\in J_1} \subset R,\ \{\beta_k\}_{k\in J_2} \subset R,$ we have:*

$$0 \le \sum_{\substack{i,j \in J_1, \\ k,l \in J_2}} \alpha_i \alpha_j \beta_k \beta_l y_{(i+j,k+l)} \le \sum_{\substack{i,j \in J_1, \\ k,l \in J_2}} \alpha_i \alpha_j \beta_k \beta_l F_2(\varphi_{i+j,k+l}).$$

The last result is the multidimensional variant of the one-dimensional case proved in [11], Theorem 3. It seems to be a corresponding form of $(c) \Leftrightarrow (b)$ from the Theorem 2.2 stated above.

Let X be the space of all absolute convergent power series in the closed polydisc $\prod_{k=1}^{n}\{|z_k|\le 1\}$, with real coefficients, and $A_1,...,A_n$ commuting selfadjoint operators acting on a Hilbert space, such that $\|A_k\|<1,\ k=1,...,n.$ Let Y be the space defined by (2), $\{B_j\}_{j\in \mathrm{N}^n}\subset Y$ and $\varepsilon>0.$ The positive cone X_+ consist of all power series with all nonnegative coefficients. One denotes

$\varphi_j(z_1,...,z_n)=z_1^{j_1}\cdots z_n^{j_n},\ \forall j=(j_1,...,j_n)\in \mathrm{N}^n.$

Theorem 3.3. *With the above notations, the following statements are equivalent:*

(a) *there exists a linear operator* $F\in L(X,Y)$ *such that*

$F(\varphi_j)=B_j,\ j\in \mathrm{N}^n,$

$\psi(A_1,...,A_n)-\varepsilon\psi(I,...,I)\le F(\psi)\le \psi(A_1,...,A_n)+\varepsilon\psi(I,...,I),\ \forall\,\psi\in X_+;$

in particular, it follows that: $\|F(\psi)\|\le\|\psi\|+\varepsilon\psi(1,...,1),\ \psi\in X_+;$

(b) *the following relations hold:*

$$A_1^{j_1}\cdots A_n^{j_n}-\varepsilon\cdot I\le B_j\le A_1^{j_1}\cdots A_n^{j_n}+\varepsilon\cdot I,\ \forall j=(j_1,...,j_n)\in \mathrm{N}^n.$$

Conclusions

The aim of the present work was to show the relationship between two apparently close topics: Mazur-Orlicz theorem (Theorem 1.5) and respectively the moment problem (Theorems 1.3, 1.4). We realize this aim by application of the above-mentioned general theorems to a few main types of concrete spaces. The reader can deduce easily the conclusion: the statements and the way of applying the two mentioned results are quite different. Approximation results are also briefly recalled, as well as their applications to the classical Markov moment problem. We considered the space of continuous functions on a product of compact subsets of R_+ and a L^1 space, with respect to a product of measures on R. These measure are not assumed to have compact support. Characterizations of the existence of the solution in terms of products of quadratic forms have been stated.

References

Akhiezer, N. I. (1965). *The Classical Moment Problem and some related Questions in Analysis.* Oliver and Boyd, Edinburgh and London.

Ambrozie, C. & Olteanu O. (2004). A sandwich theorem, the moment problem, finite-simplicial sets and some inequalities. *Revue Roumaine de Mathématiques Pures et Appliquées,* **49:** 189-210.

Berg, C., Christensen, J. P. R. & Ressel P. (1984). *Harmonic Analysis on Semigroups. Theory of Positive Definite and Related Functions*. Springer.

Berg, C. & Durán, A. (2008). The fixed point for a transformation of Hausdorff moment sequences and iteration of a rational function. *Math. Scand*. **103**:11-39.

Choquet, G. (1962). *Le problème des moments*. Séminaire d'Initiation à l'Analise, Institut H. Poincaré, Paris, France.

Cristescu, R. (1976). *Ordered Vector Spaces and Linear Operators*. Ed. Academiei, Bucharest, Romania and Abacus Press, Tunbridge Wells, Kent, England.

Fuglede, B. (1983). The multidimensional moment problem. *Expositiones Mathematicae,* Vol. I: 47-65.

C. Kleiber, Stoyanov, J. (2012). Multivariate distributions and the moment problem. *Journal of Multivariate Analysis*, **113**: 7-18.

Krein, M. G. & Nudelman, A. A., (1977). *Markov Moment Problem and Extremal Problems.* American Mathematical Society, Providence, RI.

Lemnete-Ninulescu L. (2006). Using the solution of an abstract moment problem to solve some classical complex moment problems. *Romanian Journal of Pure and Applied Mathematics*, **51:** 703-711.

Lemnete-Ninulescu, L., Olteanu, A. & Olteanu, O. (2006). Applications of the solutions of two abstract moment problems to the classical moment problem. *Mathematica*, **48(71)**, *2*:173-182.

Mihăilă, J. M., Olteanu, O. & Udrişte, C. (2007). Markov-type and operator-valued multidimensional moment problems, with some applications. *Romanian Journal of Pure and Applied Mathematics,* **52:** 405-428.

Mihăilă, J. M., Olteanu, O. & Udrişte, C. (2007). Markov-type moment problems for arbitrary compact and for some non-compact Borel subsets of R^n. *Romanian Journal of Pure and Applied Mathematics,* **52:** 655-664.

Olteanu, A., Olteanu, O. (2009). Some unexpected problems of the Moment Problem. Proc. of the Sixth Congress of Romanian Mathematicians, Ed. Academiei, Vol I: 347-355.

Olteanu, O. (1978). Convexité et prolongement d'opérateurs linéaires. *Comptes Rendus de l'Académie des Sciences, Série A,* Vol. **286**, 511-514.

Olteanu, O. (1983). Théorèmes de prolongement d'opérateurs linéaires. *Revue Roumaine de Mathématiques Pures et Appliquées.,* **28**, 10: 953-983.

Olteanu, O. (1991). Application de théorèmes de prolongement d'opérateurs linéaires au problème des moments et à une generalisation d'un théorème de Mazur-Orlicz. *Comptes Rendus de l'Académie des Sciences, Série I,* Vol. **313**, 739-742.

Olteanu, O. (1996). Applications of a general sandwich theorem for operators to the moment problem. *Romanian Journal of Pure and Applied Mathematics*, **1**, *2*: 513-521.

Olteanu, O. (2012). *Geometric aspects in operator theory and applications*. Lambert Academic Publishing, Saarbrücken, Germany.

Olteanu, O. (2013). Moment problems on bounded and unbounded domains. *International Journal of Analysis.* Volume 2013, Article ID 524264, 7 pages.

Olteanu, O. (2013) New results on Markov moment problem. *International Journal of Analysis.* Volume 2013, Article ID 901318, 17 pages.

Putinar, M. & Vasilescu, F. H. (1996). Problème des moments sur les compacts semi-algébriques, *Comptes Rendus de l'Académie des Sciences, Série I,* Vol. **323**, 787-791.

Schaefer, H. H. (1971). *Topological Vector Spaces.* Springer, New York Heidelberg Berlin.

Stoyanov J. (2004). Stieltjes classes for moment indeterminate probability distributions, *J. Applied Probab*. **41** A: 281-294.