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MAT2013-0573

Singular Sublinear Polyharmonic Problems in an Exterior Domain

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Email: info@atiner.gr URL: www.atiner.gr
URL Conference Papers Series: www.atiner.gr/papers.htm

Printed in Athens, Greece by the Athens Institute for Education and Research.

ISSN **2241-2891**
27/09/2013

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Abstract

In this paper, we prove the existence of positive continuous solution $u \in C_0(D)$, of the following m -polyharmonic singular problem involving sublinear nonlinearity: $(-\triangle)^m u = \varphi(.,u) + \psi(.,u)$ in the complementary D of the unit closed ball in $\mathbb{R}^n$, $n > 2m$. Our result improve and extend the corresponding result of [ref: YS] to the polyharmonic case .

Key words: polyharmonic elliptic equation, positive solutions, Green function, Schauder fixed point theorem

Acknowledgements: This paper is supported by NPST Program of King Saud University; project number 11-MAT1716-02.

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Introduction

The pure singular elliptic equation

$$\triangle u + p(x)u^{-\gamma} = 0,\ \gamma > 0 \text{ in } D, \qquad \text{(ref: 1.a)}$$

has been extensively studied for both bounded and unbounded domains D in $\mathbb{R}^n$ $(n \geq 3)$. We refer to ([BMZ.2, E, LS, LM, MM, MZ, Z2] and the references therein) for various existence and uniqueness results related to solutions for equation (ref: 1.a) .

In [ref: BK], Brezis and Kamin considered the following sublinear elliptic equation

$$\triangle u + q(x)u^{\alpha} = 0,\ 0 < \alpha < 1, \text{ in } \mathbb{R}^n\ (n \geq 3), \qquad \text{(ref: 1.b)}$$

and proved the existence of a unique positive solution u for (ref: 1.b) satisfying $\liminf_{|x|\to\infty} u(x) = 0$ provided that q is locally bounded such that Vq is bounded ($V = \triangle^{-1}$).

On the other hand, in [ref: YS], the authors studied the following combined elliptic problem

$$\begin{cases} \triangle u + p(x)u^{-\gamma} + q(x)u^{\alpha} = 0, \text{ in } \mathbb{R}^n \\ u(x) > 0,\ x \in \mathbb{R}^n, \\ u(x) \to 0 \text{ as } |x| \to \infty, \end{cases} \qquad \text{(ref: 1.1)}$$

where $n \geq 3$, $0 < \gamma < 1$ and $0 < \alpha < 1$ are two constants. They proved that problem (ref: 1.1) admits a solution $u^* \in C^{2+\mu}_{loc}(\mathbb{R}^n)$ provided that $p, q \in C^{\mu}_{loc}(\mathbb{R}^n)$, $0 < \mu < 1$ are nonnegative functions such that $p(x) + q(x) \neq 0$ for all $x \in \mathbb{R}^n$, and satisfying

$$\int_0^\infty t \max_{|x|=t} p(x)dt < \infty \text{ and } \int_0^\infty t \max_{|x|=t} q(x)dt < \infty. \qquad \text{(ref: 1.2)}$$

Observe that conditions (ref: 1.2) implies that the functions p and q belongs to the classical Kato class $K_n^\infty(\mathbb{R}^n)$ defined as follows .

Definition 1.1 [ref: AS, ref: Z1 – ref: Z2] . *Let* $\Omega \subseteq \mathbb{R}^n$ $(n \geq 3)$, *be an unbounded domain. A Borel measurable function* q *in* Ω *belongs to the classical Kato class* $K_n^\infty(\Omega)$ *if* q *satisfies the following conditions*

$$\lim_{r\to 0} \sup_{x\in\Omega} \int_{(|x-y|\leq r)\cap\Omega} \frac{|q(y)|}{|x-y|^{n-2}} dy = 0,$$

$$\lim_{M\to\infty} \sup_{x\in\Omega} \int_{(|y|\geq M)\cap\Omega} \frac{|q(y)|}{|x-y|^{n-2}} dy = 0.$$

In the present paper, we aim at studying the existence of positive continuous solutions for the following higher order elliptic problem

$$\begin{cases} (-\triangle)^m u = \varphi(.,u) + \psi(.,u), \text{ in } D \text{ (in the sense of distributions)} \\ u > 0, \\ \lim_{|x|\to 1} \frac{u(x)}{(|x|-1)^{m-1}} = 0, \\ u(x) \to 0 \text{ as } |x| \to \infty, \end{cases} \qquad \text{(ref: 1.3)}$$

where $D := \{x \in \mathbb{R}^n : |x| > 1\}$, m is a positive integer with $n > 2m$, and $\varphi(.,u)$ is a nonnegative singular term, while $\psi(.,u)$ is a nonnegative sublinear term satisfying some hypotheses related to the Kato class $K^{\infty}_{m,n}(D)$ (see definition 1.2 below) . In particular we improve and extend Theorem 1 in [ref: YS], to the polyharmonic case. Throughout this paper, we denote by $G^D_{m,n}$ (respectively $G^B_{m,n}$) the Green function of $(-\triangle)^m$ on D (on the unit ball B in $\mathbb{R}^n$) with Dirichlet boundary conditions $(\frac{\partial}{\partial \nu})^j u = 0$, $0 \le j \le m-1$. We recall that from [ref: B, $p.126$] (see also [ref: GS, Lemma 2.1]), we have an explicit expression for the Green function $G^B_{m,n}(\xi,\zeta)$, for $\xi,\zeta \in B$:

$$G^B_{m,n}(\xi,\zeta) = k_{m,n}|\xi-\zeta|^{2m-n}\int_1^{\frac{[\xi,\zeta]}{|\xi-\zeta|}} \frac{(\nu^2-1)^{m-1}}{\nu^{n-1}}d\nu,$$

where $k_{m,n} = \frac{\Gamma(\frac{n}{2})}{2^{2m-1}\pi^{\frac{n}{2}}[(m-1)!]^2}$ and $[\xi,\zeta]^2 = |\xi-\zeta|^2 + (1-|\xi|^2)(1-|\zeta|^2)$.

Definition 1.2 *A Borel measurable function q in D belongs to the class $K^{\infty}_{m,n}(D)$ if q satisfies the following conditions*

$$\lim_{r\to 0}\left(\sup_{x\in D}\int_{(|x-y|\le r)\cap D}\left(\frac{\rho(y)}{\rho(x)}\right)^m G^D_{m,n}(x,y)\,|q(y)|\,dy\right) = 0, \qquad (ref: 1.4)$$

$$\lim_{M\to\infty}\left(\sup_{x\in D}\int_{(|y|\ge M)}\left(\frac{\rho(y)}{\rho(x)}\right)^m G^D_{m,n}(x,y)\,|q(y)|\,dy\right) = 0, \qquad (ref: 1.5)$$

where

$$\rho(z) = \left(1-|z|^{-1}\right), \text{ for } z \in D. \qquad (ref: 1.6)$$

Note that the class $K^{\infty}_{m,n}(D)$ contains any functions q belonging to $L^s(D)\cap L^1(D)$, with $s > \frac{n}{2m}$.

In the case $m = 1$, the class $K^{\infty}_{1,n}(D)$ has been introduced and studied in [ref: BMZ.2]. In particular, it has been shown (see [ref: BMZ.2, Proposition 3.8]) that $K^{\infty}_{1,n}(D)$ properly contains the class $K^{\infty}_n(\Omega)$.

For the sake of simplicity we set $h_{m,n}$ the m -harmonic function defined in D by

$$h_{m,n}(x) := |x|^{2m-n}G^B_{m,n}(j(x),0) = k_{m,n}\int_1^{|x|}\frac{(v^2-1)^{m-1}}{v^{n-1}}dv, \qquad \text{(ref : 1.7)}$$

where $j : D \to B$ $j(x) = |x|^{-2}x$ is the inversion . We also define the potential kernel $V\phi$ by

$$V\phi(x) := V_{m,n}\phi(x) = \int_D G^D_{m,n}(x,y)\phi(y)dy, \text{ for } x \in D, \qquad \text{(ref : 1.8)}$$

and $\phi \in \mathcal{B}^+(D)$ the set of nonnegative Borel measurable functions in D.

To Study problem (ref: 1.3), we assume the following hypothesis:

(H_1) φ is a nonnegative Borel measurable function on $D\times(0,\infty)$, continuous and nonincreasing with respect to the second variable.

(H_2) $\forall$ $c > 0$, $\frac{\varphi(.,c\lambda(x))}{h_{m,n}} = q_c \in K^\infty_{m,n}(D)$, where

$$\lambda(x) = \frac{(|x|-1)^m}{|x|^{n-m}}.$$

(H_3) ψ is a nonnegative Borel measurable function on $D\times(0,\infty)$, continuous with respect to the second variable such that there exists a nontrivial nonnegative functions $p \in L^1_{loc}(D)$ and $q \in K^\infty_{m,n}(D)$ satisfying for $x \in D$ and $t > 0$,

$$p(x)h(t) \le \psi(x,t) \le q(x)h_{m,n}(x)f(t), \qquad \text{(ref : 1.9)}$$

where h is a measurable nondecreasing function on $[0,\infty)$ satisfying

$$\lim_{t\to 0^+}\frac{h(t)}{t} = \infty, \qquad \text{(ref : 1.10)}$$

and f is a nonnegative measurable function locally bounded on $[0,\infty)$ satisfying

$$\limsup_{t\to\infty}\frac{f(t)}{t} < \frac{1}{\| V(qh_{m,n})\|_\infty}. \qquad \text{(ref : 1.11)}$$

Using a fixed point argument, we prove the following main result.

Theorem 1.3 *Assume* (H_1) - (H_3). *Then the problem* (ref: 1.3) *has at least one positive continuous solution* $u \in C_0(D)$ *satisfying for each* $x \in D$

$$a\lambda(x) \le u(x) \le V\varphi(.,a\lambda(x))(x) + bV(qh_{m,n})(x) \le c\frac{(|x|-1)^m}{|x|^m},$$

where a,b *and* c *are positive constants.*

Hypotheses (H_1) - (H_3) are fulfilled with:

$$\varphi(x,t) = q(x)h_{m,n}(x)(\lambda(x))^\gamma t^{-\gamma}, \text{ for } \gamma > 0,$$

and

$$\psi(x,t) = q(x)h_{m,n}(x)t^{\alpha}, \text{ for } 0 < \alpha < 1,$$

where q is a nontrivial nonnegative function in $K^{\infty}_{m,n}(D)$.

Our plan is organized as follows. In section 2, we collect some properties of functions belonging to $K^{\infty}_{m,n}(D)$. In particular, we prove that if $s > \frac{n}{2m}$ and $\mathfrak{h} \in L^s(D)$, then for $\lambda < 2m - \frac{n}{s} < \mu$, the function

$$x \to \frac{\mathfrak{h}(x)}{|x|^{\mu-\lambda+2m}(|x|-1)^{\lambda}}, \text{ is in } K^{\infty}_{m,n}(D).$$

In section 3, we prove Theorem 1.3.

In order to simplify our statements, we define some convenient notations. We let $\mathcal{B}(D)$ the set of Borel measurable functions in D. As usual, we denote $C_0(D)$ will denote the set of continuous functions f in D vanishing continuously on ∂D and satisfying $f(x) \to 0$ as $|x| \to \infty$, within D. Note that $C_0(D)$ is a Banach spaces with respect to the uniform norm $\|u\|_{\infty} = \sup_{x\in D}|u(x)|$.

For $x,y \in D$, we let

$$\begin{cases} [x,y]^2 = |x-y|^2 + (|x|^2-1)\left(|y|^2-1\right), \\ \rho(x) = 1 - |x|^{-1}, \\ \lambda(x) = \frac{(|x|-1)^m}{|x|^{n-m}}, \\ j(x) = |x|^{-2}x. \end{cases}$$

Note that for each $(x,y) \in D^2$, we have

$$(|y|-1) \le |x|(|y|-1) \le [x,y]. \qquad \text{(ref :1.12)}$$

For any $q \in \mathcal{B}(D)$, we let

$$\|q\| := \sup_{x\in D} \int_D \left(\frac{\rho(y)}{\rho(x)}\right)^m G^D_{m,n}(x,y)|q(y)|dy. \qquad \text{(ref :1.13)}$$

Throughout this paper, for two nonnegative functions f and g defined on a set S, the notation $f(x) \approx g(x)$, $x \in S$ means that there exists $c > 0$ such that $\frac{1}{c}f(x) \le g(x) \le cf(x)$, for all $x \in S$.

We can see that

$$h_{m,n}(x) \approx \rho^m(x), \; x \in D. \qquad \text{(ref :1.14)}$$

Finally, the letter c will denote a generic positive constant which may vary from line to line.

Properties of the Green function and the Kato class $K^{\infty}_{m,n}(D)$

We collect in this section some properties of $G^{D}_{m,n}$, $(n > 2m)$ and functions belonging to the Kato class $K^{\infty}_{m,n}(D)$, which are useful for the statements of our existence results.

Proposition 2.1 [ref: BMZ]

i) *On* D^2, *we have*

$$G^{D}_{m,n}(x,y) \approx \frac{(|x|^2-1)^m(|y|^2-1)^m}{|\ x-y\ |^{n-2m}[x,y]^{2m}}. \qquad (\text{ref} : 2.1)$$

ii) *There exists* $C_{m,n} > 0$ *such that for each* $x,y,z \in D$,

$$\frac{G^{D}_{m,n}(x,z)G^{D}_{m,n}(z,y)}{G^{D}_{m,n}(x,y)} \leq C_{m,n}\left[\left(\frac{\rho(z)}{\rho(x)}\right)^m G^{D}_{m,n}(x,z) + \left(\frac{\rho(z)}{\rho(y)}\right)^m G^{D}_{m,n}(y,z)\right]. \qquad (\text{ref} : 2.2)$$

Next we give some example of functions belonging to $K^{\infty}_{m,n}(D)$.

Proposition 2.2 *Let* $s > \frac{n}{2m}$ *and* $\mathfrak{h} \in L^s(D)$. *Then for* $\lambda < 2m - \frac{n}{s} < \mu$, *the function* $\Phi(y) = \frac{\mathfrak{h}(y)}{|y|^{\mu-\lambda+2m}(|y|-1)^{\lambda}}$ *belongs to* $K^{\infty}_{m,n}(D)$.

Proof Let $s > \frac{n}{2m}$, $\lambda < 2m - \frac{n}{s} < \mu$ and $\mathfrak{h} \in L^s(D)$. First, we claim that the function Φ satisfies (ref: 1.4).

From Proposition 2.1, we deduce that

$$\left(\frac{\rho(y)}{\rho(x)}\right)^m G^{D}_{m,n}(x,y) \approx \frac{1}{|x-y|^{n-2m}}\left(\frac{|x|(|y|-1)}{[x,y]}\right)^{2m}, \text{ on } D^2. \qquad (\text{ref} : 2.3)$$

Put $\lambda^+ = \max(\lambda,0)$ and let $0 < r < 1$. Since if $|x-y| \leq r$, we have $|x| \approx |y|$, then by using (ref: 2.3) , (ref: 1.12) and the Hölder inequality , there exists a constant $c > 0$, such that

$$\begin{aligned}
&\int_{B(x,r)\cap D}\left(\frac{\rho(y)}{\rho(x)}\right)^m G^{D}_{m,n}(x,y)|\Phi(y)|dy \\
&\leq c\int_{B(x,r)\cap D}\frac{(|y|-1)^{2m-\lambda}|\mathfrak{h}(y)|}{|x-y|^{n-2m}[x,y]^{2m}|y|^{\mu-\lambda}}dy \\
&\leq c\int_{B(x,r)\cap D}\frac{(|y|-1)^{\lambda^+-\lambda}|\mathfrak{h}(y)|}{|x-y|^{n-2m+\lambda^+}|y|^{\mu-\lambda}}dy \\
&\leq c\int_{B(x,r)\cap D}\frac{|\mathfrak{h}(y)|}{|x-y|^{n-2m+\lambda^+}}dy \\
&\leq c\left(\int_D|\mathfrak{h}(y)|^s dy\right)^{\frac{1}{s}} \times \left(\int_{B(x,r)\cap D}|x-y|^{(2m-n-\lambda^+)\frac{s}{s-1}}dy\right)^{\frac{s-1}{s}} \\
&\leq cr^{2m-\frac{n}{s}-\lambda^+},
\end{aligned}$$

which tends to zero as $r \to 0$.

Now, we shall prove that Φ satisfies (ref: 1.5). Put $l = \frac{s}{s-1}$. Let $x \in D$ and $M > 1$, sufficiently large. Since for $|y| \geq M$, we have $(|y|-1) \approx |y|$, then using again (ref: 2.3), (ref: 1.12) and the Hölder inequality we obtain

$$\int_{(|y|\geq M)} \left(\frac{\rho(y)}{\rho(x)}\right)^m G_{m,n}^D(x,y)|\Phi(y)|dy$$
$$\leq c\left(\int_D |\mathfrak{h}(y)|^s dy\right)^{\frac{1}{s}} \times \left(\int_{(|y|\geq M)} \frac{1}{|x-y|^{(n-2m)l}|y|^{\mu l}} dy\right)^{\frac{1}{l}}.$$

In addition, we have

$$\int_{(|y|\geq M)} \frac{1}{|x-y|^{(n-2m)l}|y|^{\mu l}} dy$$
$$\leq c\left(\int_{(M\leq|y|\leq|x-y|)} \frac{1}{|x-y|^{(n-2m)l}|y|^{\mu l}} dy + \int_{(|y|\geq M)\cap(|x-y|\leq|y|)} \frac{1}{|x-y|^{(n-2m)l}|y|^{\mu l}} dy\right)$$
$$\leq c\left(\int_{(M\leq|y|\leq|x-y|)} \frac{1}{|y|^{(n+\mu-2m)l}} dy + \int_{(|y|\geq M)\cap(|x-y|\leq|y|)} \frac{1}{|x-y|^{(n-2m)l}|y|^{\mu l}} dy\right)$$
$$\leq c\left(\frac{1}{M^{(\mu+\frac{n}{s}-2m)l}} + \int_{(M\leq|x-y|\leq|y|)} \frac{1}{|x-y|^{(n-2m)l}|y|^{\mu l}} dy + \int_{(|x-y|\leq M\leq|y|)} \frac{1}{|x-y|^{(n-2m)l}|y|^{\mu l}} dy\right)$$
$$\leq c\left(\frac{1}{M^{(\mu+\frac{n}{s}-2m)l}} + \int_{(M\leq|x-y|\leq|y|)} \frac{1}{|x-y|^{(n+\mu-2m)l}} dy + \frac{1}{M^{\mu l}}\int_{(|x-y|\leq M\leq|y|)} \frac{1}{|x-y|^{(n-2m)l}} dy\right)$$
$$\leq c\left(\frac{1}{M^{(\mu+\frac{n}{s}-2m)l}} + \int_M^\infty \frac{1}{t^{(\mu+\frac{n}{s}-2m)l+1}} dt + \frac{1}{M^{\mu l}}\int_0^M t^{(2m-\frac{n}{s})l-1} dt\right)$$
$$\leq \frac{c}{M^{(\mu+\frac{n}{s}-2m)l}},$$

which tends to zero as $M \to \infty$.

Hence Φ satisfies (ref: 1.5) and the proof of the proposition is completed. ■

Proposition 2.3 [ref: BMZ] *Let q be a function in $K_{m,n}^\infty(D)$. Then we have*

(i) $\|q\| < \infty$.

(ii) *The function* $x \mapsto \frac{(|x|-1)^{2m}}{|x|^n} q(x)$ *is in* $L^1(D)$.

Next, we prove a sharp estimates on the potential function $V(|q|h_{m,n})$, for $q \in K_{m,n}^\infty(D)$.

Proposition 2.4 *Let $q \in K_{m,n}^\infty(D)$. Then there exists a constant $c > 0$, such that for each $x \in D$, we have*

$$\frac{1}{c}\frac{(|x|-1)^m}{|x|^{n-m}} \leq V(|q|h_{m,n})(x) \leq c\left(\frac{|x|-1}{|x|}\right)^m. \qquad \text{(ref : 2.4)}$$

Proof Since

$$G_{m,n}^D(x,y) = |x|^{2m-n}|y|^{2m-n}G_{m,n}^B(j(x),j(y)),$$

it follows that

$$\lim_{|y|\to\infty} \frac{G_{m,n}^D(z,y)}{G_{m,n}^D(x,y)} = \frac{h_{m,n}(z)}{h_{m,n}(x)}.$$

Using this fact, Fatou's lemma and (ref: 2.2), we deduce that

$$\int_D G^D_{m,n}(x,z)\frac{h_{m,n}(z)}{h_{m,n}(x)}|q(z)|dz \le \liminf_{|y|\to\infty}\int_D \frac{G^D_{m,n}(x,z)G^D_{m,n}(z,y)}{G^D_{m,n}(x,y)}|q(z)|dz$$
$$\le 2C_{m,n}\|q\|.$$

Hence the right inequality in (ref: 2.4), follows from Proposition 2.3 (i) and (ref: 1.14).

Next, we prove the left inequality in (ref: 2.4). Using Proposition 2.1 and the fact that $[x,y] \le c(|x|+1)(|y|+1)$, there exists a constant $\alpha_1 > 0$ such that for each $x,y \in D$,

$$\alpha_1\lambda(x)\lambda(y) \le G^D_{m,n}(x,y).$$

Hence

$$V(|q|h_{m,n})(x) \ge \alpha_1\lambda(x)\int_D \lambda(y)|q(y)|h_{m,n}(y)dy.$$

Thus, the required results follows from (ref: 1.14) and Proposition 2.3 (ii)

■

Proof of Theorem 1.3

We recall that $\lambda(x) = \frac{(|x|-1)^m}{|x|^{n-m}}$, for each $x \in D$.

Proof of Theorem 1.3. Assuming (H_1) - (H_3), we shall use the Schauder fixed point theorem. Let K be a compact of D such that

$$0 < \alpha := \int_K \lambda(y)p(y)dy < \infty.$$

We put $\beta := \min\{\lambda(x) : x \in K\}$. As in the proof of Proposition 2.4, there exists a constant $\alpha_1 > 0$ such that for each $x,y \in D$,

$$\alpha_1\lambda(x)\lambda(y) \le G^D_{m,n}(x,y). \qquad (\text{ref}: 3.1)$$

From (ref: 1.10), we deduce that there exists $a > 0$ such that

$$\alpha_1\alpha h(a\beta) \ge a. \qquad (\text{ref}: 3.2)$$

Using (ref: 2.4) , there exists a constant $\alpha_2 > 0$ such that

$$\alpha_2\lambda(x) \le V(qh_{m,n})(x),\ \forall x \in D. \qquad (\text{ref}: 3.3)$$

On the other hand, since $q \in K^\infty_{m,n}(D)$, then by Proposition 2.4, we have $\|V(qh_{m,n})\|_\infty < \infty$. So taking $\limsup_{t\to\infty}\frac{f(t)}{t} < \delta < \frac{1}{\|V(qh_{m,n})\|_\infty}$, we deduce that there exists $\rho > 0$ such that for $t \geq \rho$ we have $f(t) \leq \delta t$. Put $\gamma = \sup_{0 \leq t \leq \rho} f(t)$. So we have that

$$0 \leq f(t) \leq \delta t + \gamma,\ t \geq 0. \qquad (\text{ref} : 3.3a)$$

Furthermore, from (H_2) and Proposition 2.4, we have that $\| V(\varphi(.,a\lambda)) \|_\infty < \infty$. Put

$$\chi = \delta(\| V\varphi(.,a\lambda) \|_\infty + b \| V(qh_{m,n}) \|_\infty) + \gamma$$

and

$$b = \max\{\frac{a}{\alpha_2}, \frac{\delta \| V\varphi(.,a\lambda) \|_\infty + \gamma}{1 - \delta \| V(qh_{m,n}) \|_\infty}\}. \qquad (\text{ref} : 3.4)$$

Let

$$\Lambda = \{u \in C_0(D) : a\lambda(x) \leq u(x) \leq V\varphi(.,a\lambda(x))(x) + bV(qh_{m,n})(x),\ \forall x \in D\}.$$

Then Λ is a nonempty closed bounded and convex set in $C_0(D)$. We define the integral operator T on Λ by

$$Tu(x) = \int_D G^D_{m,n}(x,y)[\varphi(y,u(y)) + \psi(y,u(y))]dy,\ \forall x \in D.$$

We claim that $T\Lambda$ is relatively compact in $C_0(D)$ and $T\Lambda \subset \Lambda$.

First we prove the equicontinuity of $T(\Lambda)$ on $\overline{D} \cup \{\infty\}$.

From the hypotheses and (ref: 3.3a), we have that for each $u \in \Lambda$,

$$\varphi(.,u) \leq \varphi(.,a\lambda) = q_a h_{m,n} \text{ and } \psi(.,u) \leq \chi q h_{m,n}. \qquad (\text{ref} : 3.5)$$

Let $c_0 > 0$, such that

$$\| h_{m,n} \|_\infty \leq c_0.$$

Let $x_0 \in \overline{D}$ and $\varepsilon > 0$. Since $\tilde{q} := q_a + \chi q \in K^\infty_{m,n}(D)$, then by [ref: BMZ, Proposition 4.2] there exist $r > 0$ and $M > 1$ such that

$$\sup_{z \in D} \frac{1}{h_{m,n}(z)} \int_{B(x_0,2r)\cap D} G^D_{m,n}(z,y) h_{m,n}(y) |\tilde{q}(y)| dy \leq \frac{\varepsilon}{4c_0} \qquad (\text{ref} : 3.6)$$

and

$$\sup_{z \in D} \frac{1}{h_{m,n}(z)} \int_{(|y| \geq M)} G^D_{m,n}(z,y) h_{m,n}(y) |\tilde{q}(y)| dy \leq \frac{\varepsilon}{4c_0}, \qquad (\text{ref} : 3.7)$$

where $B(x_0,r)$ is the open ball of center x_0 and radius r.

For each $x,x' \in B(x_0,r) \cap D$ and $u \in \Lambda$, we have $u \in \Lambda$,

$$|Tu(x) - Tu(x')| \leq \int_D |G^D_{m,n}(x,y) - G^D_{m,n}(x',y)| h_{m,n}(y) |\tilde{q}(y)| dy$$
$$\leq 2c_0 \sup_{z \in D} \frac{1}{h_{m,n}(z)} \int_{B(x_0,2r)\cap D} G^D_{m,n}(z,y) h_{m,n}(y) |\tilde{q}(y)| dy$$
$$+ 2c_0 \sup_{z \in D} \frac{1}{h_{m,n}(z)} \int_{\Omega_2} G^D_{m,n}(z,y) h_{m,n}(y) |\tilde{q}(y)| dy$$

$$+\int_{\Omega_1} |G^D_{m,n}(x,y) - G^D_{m,n}(x',y)|h_{m,n}(y)|\widetilde{q}(y)|dy,$$

where $\Omega_1 = (|x_0 - y| \geq 2r) \cap (1 < |y| \leq M)$
and $\Omega_2 = (|x_0 - y| \geq 2r) \cap (|y| \geq M).\Omega$

Hence

$$|Tu(x) - Tu(x')| \leq \varepsilon + \int_{\Omega_1} |G^D_{m,n}(x,y) - G^D_{m,n}(x',y)|h_{m,n}(y)|\widetilde{q}(y)|dy.$$

On the other hand, using (ref: 2.1), there exists a constant $c > 0$ such that for every $y \in \Omega_1$ and $x \in B(x_0, r) \cap D$,

$$G^D_{m,n}(x,y) \leq c\frac{(\rho(x)\rho(y))^m}{|\ x-y\ |^{n-2m}}.$$

So we deduce that

$$\begin{aligned}
|G^D_{m,n}(x,y) - G^D_{m,n}(x',y)| &\leq G^D_{m,n}(x,y) + G^D_{m,n}(x',y) \\
&\leq c\left[\frac{(\rho(x))^m(\rho(y))^m}{|x-y|^{n-2m}} + \frac{(\rho(x'))^m(\rho(y))^m}{|x'-y|^{n-2m}}\right] \\
&\leq c\left[\frac{1}{|x-y|^{n-2m}} + \frac{1}{|x'-y|^{n-2m}}\right](\rho(y))^m \\
&\leq c(|y| - 1)^m \\
&\leq c\frac{(|y|-1)^m}{|y|^{n-m}}.
\end{aligned}$$

Now since $G^D_{m,n}$ is continuous outside the diagonal,we deduce by (ref: 1.14) ,

Proposition 2.3 (ii) and the dominated convergence theorem that

$$\int_{\Omega_1} |G^D_{m,n}(x,y) - G^D_{m,n}(x',y)|h_{m,n}(y)|\widetilde{q}(y)|dy \to 0 \text{ as } |x - x'| \to 0.$$

Hence $|Tu(x) - Tu(x')| \to 0$ as $|x - x'| \to 0$ uniformly for all $u \in \Lambda$.

To establish compactness we claim that $\lim_{|x|\to\infty} |Tu(x)| = 0,$ uniformly for all $u \in \Lambda$.

Indeed, Let $M > 1$ and $x \in D$ such that $|x| \geq M + 1$. Then from (ref: 3.7), we deduce that (ref: 3.7),

$$|Tu(x)| \leq \frac{\varepsilon}{4} + \int_{(1<|y|\leq M)} G^D_{m,n}(x,y)h_{m,n}(y)|\widetilde{q}(y)|dy$$

Since for $y \in B(0,M) \cap D$, we have $|x - y| \geq 1$, then by (ref: 2.1) , we get (ref: 2.1)

$$\begin{aligned}
|Tu(x)| &\leq \frac{\varepsilon}{4} + c\int_{(1<|y|\leq M)} \frac{(\rho(y))^m}{|x-y|^{n-2m}} h_{m,n}(y)|\widetilde{q}(y)|dy \\
&\leq \frac{\varepsilon}{4} + \frac{c}{(|x| - M)^{n-2m}} \int_{(1<|y|\leq M)} (|y| - 1)^m h_{m,n}(y)|\widetilde{q}(y)|dy \\
&\leq \frac{\varepsilon}{4} + \frac{c}{(|x| - M)^{n-2m}} \int_{(1<|y|\leq M)} \frac{(|y|-1)^m}{|y|^{n-m}} h_{m,n}(y)|\widetilde{q}(y)|dy.
\end{aligned}$$

Again using (ref: 1.14) and Proposition 2.3 (ii), we obtain $\lim_{|x|\to\infty} |Tu(x)| = 0,$ uniformly for all $u \in \Lambda.$

Moreover, the family $T\Lambda$ is uniformly bounded. It follows by Ascoli's theorem that $T\Lambda$ is relatively compact in $C_0(D).$

Next, we prove that $T\Lambda \subset \Lambda.$

Indeed, let $u \in \Lambda$ and $x \in D$, then from the hypotheses and (ref: 3.4) , we have

$$\begin{aligned} Tu(x) &\leq V\varphi(.\,,a\lambda(x))(x) + \int_D G^D_{m,n}(x,y)q(y)h_{m,n}(y)f(u(y))dy \\ &\leq V\varphi(.\,,a\lambda)(x) + \int_D G^D_{m,n}(x,y)q(y)h_{m,n}(y)[\delta u(y) + \gamma]dy \\ &\leq V\varphi(.\,,a\lambda)(x) + \chi\int_D G^D_{m,n}(x,y)q(y)h_{m,n}(y)dy \\ &\leq V\varphi(.\,,a\lambda)(x) + bV(qh_{m,n})(x). \end{aligned}$$

Moreover from the monotonicity of $h,$ (ref: 3.1) and (ref: 3.2) , we have

$$\begin{aligned} Tu(x) &\geq \int_D G^D_{m,n}(x,y)\psi(y,u(y))dy \\ &\geq \alpha_1\lambda(x)\int_D \lambda(y)p(y)h(a\lambda(y))dy \\ &\geq \alpha_1\lambda(x)h(a\beta)\int_K \lambda(y)p(y)dy \\ &\geq a\lambda(x). \end{aligned}$$

So $T\Lambda \subset \Lambda.$

We claim that, T is continuous . To this end, we consider a sequence $(u_k)_k$ in $\Lambda,$ which converges uniformly to a function u in $\Lambda.$ Since φ and ψ are continuous with respect to the second variable, we deduce by (ref: 3.5) , Proposition 2.4 and the dominated convergence theorem that

$$\forall x \in D\ ,\ Tu_k(x) \to Tu(x) \text{ as } k \to \infty.$$

Since $T\Lambda$ is relatively compact in $C_0(D),$ then we have the uniform convergence. Hence T is a compact operator mapping from Λ to itself. So the Schauder fixed point theorem leads to the existence of a function $u \in \Lambda$ such that

$$u(x) = \int_D G^D_{m,n}(x,y)[\varphi(y,u(y)) + \psi(y,u(y))]dy,\ \forall x \in D. \qquad \text{(ref : 3.8)}$$

Finally, using (ref: 3.8), (ref: 3.5), (ref: 1.14) and Proposition 2.3 (ii), one can check that u is the required solution. ■

Example *Let* $\gamma > 0,$ $0 < \alpha < 1$ *and* $q \in K^\infty_{m,n}(D)$ *with* $q \gneqq 0.$ *Then the problem*

$$\begin{cases} (-\triangle)^m u = q(x)h_{m,n}(x)[(\lambda(x))^\gamma u^{-\gamma} + u^\alpha], \text{ in } D \\ u > 0 \\ \lim_{|x|\to 1} \frac{u(x)}{(|x|-1)^{m-1}} = 0, \\ u(x) \to 0 \text{ as } |x| \to \infty, \end{cases}$$

has at least one positive continuous solution $u \in C_0(D)$ *satisfying*

$$\frac{1}{c}\,\frac{(|x|-1)^m}{|x|^{n-m}} \le u(x) \le c\,\frac{(|x|-1)^m}{|x|^m},$$

where c *is a positive constant.*

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