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Some insights about *PL*(7, 2) Codes

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Abstract

The research on codes in Lee metric has increased in last decades due to their several applications. The interest in these codes is mostly centered in their existence and enumeration. A well known conjecture of Golomb and Welch states that for $n > 2$ and $r > 1$ there are no perfect r-error correcting Lee codes of word length n over Z, shortly $PL(n, r)$ codes. Although many efforts have been made, the conjecture is still far from being solved. It seems that the most difficult cases of the conjecture are those in which $r = 2$. In this paper we give a contribution for the proof of the nonexistence of $PL(n, 2)$ codes, in particular, for the proof of the nonexistence of $PL(7, 2)$ codes. We present some results, based on the assumption that there exist such codes, and give a strategy which we believe that will be helpful to prove the nonexistence of $PL(7, 2)$ codes.

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.Introduction

Geometric problems such as '*for what values of n and r does the n-dimensional sphere of radius r tile n-dimensional space?'* are popular, mainly, by their applications on real-life problems. For example, n-space tilings by certain subsets constitute different types of error correcting codes, see [4].

The most common metric in coding theory is the Hamming metric, however we are interested on another frequently used metric, the Lee metric. Since its first application, related with signal transmission over noisy channels, see [9] and [14], many studies dealing with the Lee metric have appeared, in particular, studies involving different types of codes in the Lee metric. In fact, the interest in Lee codes has been increasing due to their several applications, see e.g. [1], [2], [3] and [11]. In this paper we focus our attention on perfect error correcting Lee codes, introduced in [4].

Let (Z^n, ρ_L) be a metric space, where Z is the set of all integer numbers, n is a positive integer number and ρ_L denotes the Lee metric, $\mathrm{M} \subset \mathrm{Z}^n$ is a *perfect r-error correcting Lee code of word length n over* Z if all spheres of radius r centered at the elements of M form a partition of Z^n.

The existence and enumeration of these codes are central problems in the area of Lee codes. It is proved that there are such codes for $n \geq 1$ and $r = 1$ as well as for $n \leq 2$ and $r \geq 1$, see [4]. However, it seems that there are no perfect error correcting Lee codes for other values of n and r. This conjecture is known as the Golomb - Welch conjecture [4]:

Conjecture 1. There are no perfect r-error correcting Lee codes of word length n over Z for $n > 2$ and $r > 1$.

As there exist perfect 1-error correcting Lee codes of word length n over Z for all $n \geq 1$, perhaps the most difficult cases of the Golomb-Welch conjecture are those in which $r = 2$.

Although there is an extensive literature on the subject, the conjecture is still far from being solved. Several authors have applied distinct methods to prove the nonexistence of perfect Lee codes for certain values of n and r, however, in general, it is not possible to extend them to other values of the parameters to settle the conjecture completely.

Golomb and Welch have shown in [4] that for $n > 2$ there exists r_n so that for all $r > r_n$ there is no perfect r-error correcting Lee code of word length n over Z, not being specified the value of r_n. In [5] it is proved the Golomb-Welch conjecture for $n = 3$ and $r > 1$. Using computational resources Spacapan [12] showed that there are no perfect Lee codes for $n = 4$ and $r > 1$. Horak has shown in [7] and [6], respectively, that there are no perfect Lee codes for $3 \leq n \leq 5$ and $r > 1$, and for $n = 6$ and $r = 2$.

Unfortunately, it seems that the known proofs do not bring out results and techniques which allow us to skip to higher dimensions. The difficulty in proving the conjecture for other values of the parameters has led some authors to consider special types of perfect Lee codes. Actually, it would be of great help to find out results for these types of Lee codes, using different approaches, having in mind their extension to perfect Lee codes. Next, we present some interesting results.

In [10], considering metric spaces (Z_q^n, ρ_L), Post showed that there are no periodic perfect Lee codes for $3 \le n \le 5$, $r \ge n-1$ and $q \ge 2r+1$, and for $n \ge 6$, $r \ge \frac{\sqrt{2}}{2}n - \frac{1}{4}(3\sqrt{2}+2)$ and $q \ge 2r+1$. These results were improved by Spacapan in [13], where it is proved that there is no periodic perfect Lee code for $r \ge n \ge 3$.

Horak and Grosek [8] have recently proved, using a new approach, the nonexistence of linear r-error correcting perfect Lee codes for $r = 2$ and $7 \le n \le 11$.

Since, for $r = 2$ the Golomb-Welch conjecture is proved for few values of n, our interest is focused on the proof of the nonexistence of perfect 2-error correcting Lee codes of word length n over Z for $n \ge 7$. In particular, we are interested on proving the nonexistence of perfect 2-error correcting Lee codes of word length 7 over Z.

In this paper we present a possible strategy to prove the nonexistence of these codes. Assuming that there exist such codes, we state some results which condition their existence.

In section 2 we introduce some definitions and notation. In section 3 we present the method which we believe may be applied to prove the nonexistence of such codes and the first results obtained so far.

2. Definitions and notation

Let (C, ρ) be a metric space, where C is a set and ρ a metric. Any subset M of C, $|\mathrm{M}| \ge 2$, is called *code*. The elements of C are referred as *words*, in particular, the elements of M will be called *codewords*.

A sphere centered at $W \in C$ with radius r is denoted by $S(W, r)$ and is defined as follows

$$S(W, r) = \{V \in C : \rho(V, W) \le r\}.$$

If $W \in \mathrm{M}$ and $V \in S(W, r)$, with $V \ne W$, we say that the codeword W covers the word V.

A code M is a *r-error correcting code* if for any two codewords $W, V \in \mathrm{M}$ it holds $S(W, r) \cap S(V, r) = \varnothing$. If, in addition, $\bigcup_{W \in M} S(W, r) = C$, then M is a *perfect r-error correcting code*.

In this paper we consider codes in the metric spaces (Z^n, ρ_L), where Z^n is the n-fold Cartesian product of Z, being Z the set of integer numbers, and ρ_L denotes the Lee metric. Considering $W, V \in \mathrm{Z}^n$, with $W = (w_1, w_2, \ldots, w_n)$ and $V = (v_1, v_2, \ldots, v_n)$, $\rho_L(W, V)$ is given by $\rho_L(W,V) = \sum_{i=1}^{n} |w_i - v_i|$. In these metric spaces if $\mathrm{M} \subset \mathrm{Z}^n$ is a perfect r-error correcting code then M is called *perfect r-*

error correcting Lee code of word length n over Z and is called, shortly, by $PL(n, r)$ code. We note that, given $W, V \in Z^n$ and $r > 0$, $S(W, r) \cap S(V, r) = \varnothing$ if and only if $\rho_L(W, V) \geq 2r + 1$.

Let $M \subset Z^n$ be a $PL(n, r)$ code. Then, $\{S(W, r): W \in M\}$ can be seen as a partition of Z^n. On the other hand, given a codeword $W \in M$, the set of unit cubes centered at V, with $V \in S(W, r)$, tiles R^n. We recall that a unit cube centered at $V = (v_1, v_2, \ldots, v_n) \in Z^n$ is the set

$$\{X = (x_1, x_2, ..., x_n) \in R^n : x_i = v_i + \alpha_i, -1/2 \leq \alpha_i \leq 1/2\}.$$

We are interested into analyze the existence of $PL(n, 2)$ codes for $n \geq 7$, in particular, $PL(7, 2)$ codes. We will deal with $PL(n, 2)$ codes following a strategy introduced by Horak [6] for proving the nonexistence of $PL(n, 2)$ codes for $3 \leq n \leq 6$. In this context we will assume, by contradiction, that there exists a $PL(n, 2)$ code M for $n \geq 7$ and also that $O = (0, 0, \ldots, 0) \in M$, we will focus our attention on the subset of M containing the codewords which cover all words $W \in Z^n$ satisfying $\rho_L(W, O) = 3$. Note that, O cover all words W such that $\rho_L(W, O) \leq 2$. Our aim is to prove that it is not possible to cover all words $W \in Z^n$ such that $\rho_L(W, O) = 3$, without superposition between codewords of M.

Consider $W \in Z^n$ such that $\rho_L(W, O) = 3$, then $W = (w_1, w_2, \ldots, w_n)$ is of one and only one of the three distinct types: $[\pm 3]$, if there exists a unique element $i \in \{1, 2, \ldots, n\}$such that $w_i \neq 0$ and $|w_i|=3$; $[\pm 2, \pm 1]$, if there are exactly two elements $i, j \in \{1, 2, \ldots, n\}$ such that $w_i, w_j \neq 0$, $|w_i| = 2$ and $|w_j| = 1$; $[\pm 1^3]$, if there are exactly three elements $i, j, k \in \{1, 2, \ldots, n\}$ such that $w_i, w_j, w_k \neq 0$ and $|w_i| = |w_j| = |w_k| = 1$.

Let $T \subset M$ be the set of codewords which cover all words $W \in Z^n$ such that $\rho_L(W, O) = 3$. Then, $T \in T$ satisfies $\rho_L(T, O) = 5$. In fact, $\rho_L(T, O) \geq 5$ since $r = 2$, and supposing $\rho_L(T, O) \geq 6$ we have $\rho_L(T, W) \geq 3$ for all word $W \in Z^n$ satisfying $\rho_L(W, O) = 3$, that is, the codeword T does not cover W. Thus, $T \in T$ is of one and only one of the types: $[\pm 5]$, $[\pm 4, \pm 1]$, $[\pm 3, \pm 2]$, $[\pm 3, \pm 1^2]$, $[\pm 2^2, \pm 1]$, $[\pm 2, \pm 1^3]$ and $[\pm 1^5]$. The subsets of T containing codewords of these types will be denoted, respectively, by A, B, C, D, E, F and G, and we set $a = |A|$, $b = |B|$, $c = |C|$, $d = |D|$, $e = |E|$, $f = |F|$ and $g = |G|$, where $|X|$ denotes the cardinality of X.

We will consider $I = \{+1, +2, \ldots, +n, -1, -2, \ldots, -n\}$ as the set of signed coordinates. Let $W = (w_1, w_2, \ldots, w_n)$, $V = (v_1, v_2, \ldots, v_n) \in Z^n$ and $i \in I$, if $iw_{|i|}>0$ and $iv_{|i|}>0$, then we say that the $|i|$-th coordinates of W and V have the sign of i and, therefore, W and V are sign equivalent in the $|i|$-th coordinate.

Let $H \subset Z^n$ and $W = (w_1, w_2, \ldots, w_n) \in H$. Consider $i, j \in I$, with $|i| \neq |j|$, and k a positive integer number. H_i, H_{ij} and $H_i^{(k)}$ will denote, respectively:

- $H_i = \{W \in H: iw_{|i|} > 0\}$;
- $H_{ij} = \{W \in H: iw_{|i|} > 0 \wedge jw_{|j|} > 0\}$;
- $H_i^{(k)} = \{W \in H: iw_{|i|} > 0 \wedge w_{|i|} = \pm k\}$.

Our aim is to characterize the set T, having in mind to prove that any characterization of T contradicts the definition of $PL(n, 2)$ code. In next section we will analyze A, B, C, D, E, F, $G \subset$ T through the study of the cardinality of their subsets of codewords which are sign equivalent in some coordinates.

3. Conditions for the existence of *PL*(*n*, 2) codes

Our intention is to prove the nonexistence of $PL(n, 2)$ codes for values $n \geq 7$. However, it is not easy to prove the Golomb-Welch conjecture when n is big and $r = 2$. As the nonexistence of $PL(n, 2)$ codes it is already proved for $3 \leq n \leq 6$, we are initially interested on proving the nonexistence of $PL(7, 2)$ codes. Although we have not yet stablished the nonexistence of these codes, in this section we present some conditions which have to be satisfied by $PL(n, 2)$ codes and which we hope to be useful in the proof of the nonexistence of such codes. We begin deriving results for $PL(n, 2)$ codes and then, in the last part of this section, we present specific results for $PL(7, 2)$ codes.

3.1. *PL(n, 2) codes*

Suppose that there exists a $PL(n, 2)$ code M $\subset$ Z^n for $n \geq 7$. As stated before, we assume that $O = (0, 0, \ldots, 0)$ is a codeword of M and so all words $W \in$ Z^n such that $\rho_L(W, O) \leq 2$ are covered by O. The words $W \in$ Z^n which verify $\rho_L(W, O) = 3$ are not covered by O and our interest is to characterize the set T $\subset$ M composed by the codewords which cover all these words. We are going to focus our attention on the subsets A, B, C, D, E, F and G of T which contain, respectively, codewords of the types $[\pm 5]$, $[\pm 4, \pm 1]$, $[\pm 3, \pm 2]$, $[\pm 3, \pm 1^2]$, $[\pm 2^2, \pm 1]$, $[\pm 2, \pm 1^3]$ and $[\pm 1^5]$.

We begin by stating some preliminary results proved by Horak [6], which condition the cardinality of the subsets of T.

Proposition 2: The parameters a, b, c, d, e, f and g must satisfy the following system of equations:

$$\begin{cases} a + b + c + d = 2n \\ b + 2c + 2d + 4e + 3f = 8\,C_2^n \\ d + e + 4f + 10g = 8\,C_3^n\,. \end{cases}$$

Lemma 3. For each $i \in$ I, $|D_i \cup E_i| + 3|F_i| + 6|G_i| = 4C_2^{n-1}$.

Lemma 4. For each $i, j \in$ I, $|i| \neq |j|$, $|D_{ij} \cup E_{ij}| + 2|F_{ij}| + 3|G_{ij}| = 2(n - 2)$.

Lemma 5. For $i \in$ I, $|F_i| \leq 2(n-1) - (|D_i^{(1)}| + |E_i| + |E_i^{(1)}|) + \left\lfloor \frac{2(n-1) - 2(|D_i^{(3)}| + |E_i^{(2)}|)}{3} \right\rfloor$.

We note that $\lfloor x \rfloor$ denotes the highest integer number less or equal to x.

It is clear that there exist many nonnegative integer solutions for the system of equations from Proposition 2. However, we are interested in finding out the 'good' solutions, that is, solutions which do not contradict the definition of $PL(n, 2)$ codes. Here we provide insights towards the characterization of 'good' solutions.

The cardinality of any of the sets A, B, C, D, E, F and G is influenced by the cardinality of their index subsets. For example, considering the set G we can relate $|G|$ with the cardinality of all subsets G_i, $i \in \mathrm{I}$. In fact,

$$|G| = \frac{1}{5}\sum_{i\in I}|G_i|.$$

Besides, for any $i \in \mathrm{I}$

$$|G_i| = \frac{1}{4}\sum_{j\in I\setminus\{i,-i\}}|G_{ij}|.$$

We can derive equivalent equalities when we consider the other sets of T.

The codewords of F and G have more nonzero coordinates when compared with any other codewords of T. In particular, considering the system of equations from Proposition 2, g is the variable with highest coefficient. Thus, we will give particular attention to these sets.

The following lemma restricts the variability of $|F_i|$.

Lemma 6. For each $i \in \mathrm{I}$, $|F_i| \le \frac{8(n-1)+1}{3} - |D_i| - \frac{5}{3}|E_i|$.

Proof. Let $i \in \mathrm{I}$. By Lemma 5 we know that

$$|F_i| \le 2(n-1) - (|D_i^{(1)}| + |E_i| + |E_i^{(1)}|) + \left\lfloor \frac{2(n-1)-2(|D_i^{(3)}|+|E_i^{(2)}|)}{3} \right\rfloor.$$

Thus,

$$|F_i| \le 2(n-1) - (|D_i^{(1)}| + |E_i| + |E_i^{(1)}|) + \frac{2(n-1)-2(|D_i^{(3)}|+|E_i^{(2)}|)}{3}$$

and, equivalently,

$$|F_i| \le \frac{8(n-1)}{3} - \left(|D_i^{(1)}| + \frac{2}{3}|D_i^{(3)}|\right) - \left(|E_i| + |E_i^{(1)}| + \frac{2}{3}|E_i^{(2)}|\right).$$

As the codewords of D are of type $[\pm 3, \pm 1^2]$ and the codewords of E are of type $[\pm 2^2, \pm 1]$, then $|D_i|=|D_i^{(1)}|+|D_i^{(3)}|$ and $|E_i|=|E_i^{(1)}|+|E_i^{(2)}|$. Therefore,

$$|F_i| \le \frac{8(n-1)}{3} - \left(|D_i| - \frac{1}{3}|D_i^{(3)}|\right) - \left(2|E_i| - \frac{1}{3}|E_i^{(2)}|\right).$$

We note that $0 \le |D_i^{(3)}| \le 1$, otherwise, there are $W, V \in D_i^{(3)}$, $W \ne V$, covering a same word of type $[\pm 3]$. In fact, W and V would cover a same word U with the $|i|$-th coordinate $u_{|i|}$ satisfying $iu_{|i|}>0$ and $u_{|i|} = \pm 3$. We also note that $|E_i| \ge |E_i^{(2)}|$. Consequently,

$$|F_i| \le \frac{8(n-1)}{3} - |D_i| + \frac{1}{3} - 2|E_i| + \frac{1}{3}|E_i| = \frac{8(n-1)+1}{3} - |D_i| - \frac{5}{3}|E_i|. \square$$

The next two lemmas establish similar results for $|G_i|$.

Lemma 7. For each $i \in$ I, $|G_i| \geq \frac{|D_i \cup E_i| + (n-1)(n-6)|}{3} - \frac{1}{6}$.

Proof. From Lemma 3 we get $|D_i \cup E_i| + 3|F_i| + 6|G_i| = 4C_2^{n-1}$ for all $i \in$ I. Therefore,

$$|F_i| = \frac{2(n-1)(n-2) - 6|G_i| - |D_i \cup E_i|}{3}$$

for all $i \in$ I. Considering Lemma 6, it follows that

$$\frac{2(n-1)(n-2) - 6|G_i| - |D_i \cup E_i|}{3} \leq \frac{8(n-1)+1}{3} - |D_i| - \frac{5}{3}|E_i| \leq \frac{8(n-1)+1}{3} - |D_i \cup E_i|$$

.

Thus,

$$|G_i| \geq \frac{|D_i \cup E_i| + (n-1)(n-6)}{3} - \frac{1}{6}$$

for all $i \in$ I. □

Lemma 8. For each $i \in$I, $|G_i| \leq \frac{(n-1)(n-2)}{3}$. In particular, if $n \equiv 0(\mathit{mod}\ 3)$, then $|G_i| \leq \frac{(n-1)(n-3)}{3}$. If $n \equiv 1(\mathit{mod}\ 3)$, then $|G_i| \leq \frac{(n-1)(2n-5)}{6}$.

Proof. By Lemma 3 we know that $6|G_i| \leq 4C_2^{n-1}$ for all $i \in$ I. Equivalently,

$$|G_i| \leq \frac{(n-1)(n-2)}{3}$$

for $i \in$ I.

From Lemma 4 we get

$$|G_{ij}| \leq \frac{2(n-2)}{3}, \tag{1}$$

where $i, j \in$ I and $|i| \neq |j|$.

If $n \equiv 0(\mathit{mod}\ 3)$, then there is a positive integer number k so that $n = 3k$. Therefore, (1) assumes the form

$$|G_{ij}| \leq \frac{2(3k-2)}{3} = 2k - 1 - \frac{1}{3}.$$

As $|G_{ij}|$ is a nonnegative integer number, it follows that $|G_{ij}| \leq 2k-2$. Then, taking into account that $k = \frac{n}{3}$, we get

$$|G_{ij}| \leq 2\left(\frac{n}{3} - 1\right). \tag{2}$$

The codewords of G are of type $[\pm 1^5]$. Therefore, for $i \in$ I, we have

$$|G_i| = \frac{1}{4} \sum_{j \in I \setminus \{i,-i\}} |G_{ij}|.$$

As $|I \setminus \{i,-i\}| = 2(n-1)$, taking into account (2) it follows that

$$|G_i| \leq \frac{1}{4} \times 2(n-1) \times 2\left(\frac{n}{3} - 1\right) = \frac{(n-1)(n-3)}{3}$$

for $i \in$ I.

If $n \equiv 1(mod\ 3)$, then $n = 3k + 1$, where k is a positive integer number. In these conditions, from (1) we get

$$|G_{ij}| \leq 2k - \frac{2}{3}.$$

As $|G_{ij}|$ is a nonnegative integer number, $|G_{ij}| \leq 2k - 1$. Taking into account that $k = \frac{n-1}{3}$, it follows that

$$|G_{ij}| \leq 2\left(\frac{n-1}{3}\right) - 1 = \frac{2n-5}{3}.$$

Thus, for each $i \in$ I we have

$$|G_i| = \frac{1}{4} \sum_{j \in I \setminus \{i,-i\}} |G_{ij}| \leq \frac{1}{4} \times 2(n-1) \times \frac{2n-5}{3} = \frac{(n-1)(2n-5)}{6}. \ \square$$

3.1.1. *PL(7, 2) codes*

In this section we introduce a possible strategy to prove the nonexistence of *PL*(7, 2) codes. We believe that next results will be helpful to build the complete proof. We are going to present only the first steps and the general idea about the method to be applied to prove it.

Consider the system of equations from Proposition 2. For $n = 7$ we get

$$\begin{cases} a + b + c + d = 14 \\ b + 2c + 2d + 4e + 3f = 168 \\ d + e + 4f + 10g = 280. \end{cases} \tag{3}$$

Taking into account the previous results we can restrict the variation of g limiting the variation of $|G_i|$, where $i \in$ I.

Lemma 9. For each $i \in$ I, $3 \leq |G_i| \leq 9$.

Proof. From Lemma 7 we get

$$|G_i| \geq \frac{|D_i \cup E_i| + 6}{3} - \frac{1}{6} \geq \frac{6}{3} - \frac{1}{6} = \frac{11}{6} > 1$$

for $i \in$ I.

Suppose, by contradiction, that there is an $i \in$ I such that $|G_i| = 2$. Thus, by Lemma 7, we have $2 \geq \frac{|D_i \cup E_i| + 6}{3} - \frac{1}{6}$ and, consequently, $|D_i \cup E_i| = 0$. Let $W, V \in G_i$, with $W \neq V$. At most, there exists one element $j \in$ I\{i, -i}, such that, $W, V \in G_{ij}$, otherwise, if there are $j, k \in$ I\{i, -i}, with $|j| \neq |k|$, and $W, V \in G_{ijk}$, then W and V cover a same word U of type $[\pm 1^3]$, whose $|i|$-th, $|j|$-th and $|k|$-th coordinates satisfy $iu_{|i|}, ju_{|j|}, ku_{|k|} > 0$, which contradicts the definition of perfect Lee codes. As the codewords of G are of type $[\pm 1^5]$, there are, at least, six elements $k \in I \backslash \{i, -i\}$, distinct, such that $|G_{ik}| = 1$. Then, by Lemma 4, $|D_{ik} \cup E_{ik}| + 2|F_{ik}| = 7$ and, therefore, $|D_{ik} \cup E_{ik}| > 0$ for all these elements k. Consequently, $|D_i \cup E_i|>0$, which is a contradiction. Thus, we have $|G_i| \geq 3$ for all $i \in I$.

Considering now Lemma 8, as $7 \equiv 1(mod\ 3)$, we get $|G_i| \leq 9$ for all $i \in I$.

Therefore, for each $i \in I$ we have $3 \leq |G_i| \leq 9$. □

The restriction of the range of the values for $|G_i|$ reduces the possible values for g. Thus, our strategy consists in proving that for each integer number x satisfying $3 \leq x \leq 9$, the hypothesis $|G_i| = x$ implies a contradiction on the definition of $PL(7, 2)$ code, as we will see in the next lemma for the case $|G_i| = 9$.

Lemma 10. For all $i \in I$, $|G_i| \neq 9$.

Proof. Let $i \in I$. Suppose, by contradiction, that $|G_i| = 9$. As $|G_i| = \frac{1}{4} \sum_{j \in I \backslash \{i,-i\}} |G_{ij}|$, then $\sum_{j \in I \backslash \{i,-i\}} |G_{ij}| = 36$. From Lemma 4 we have $|D_{ij} \cup E_{ij}| + 2|F_{ij}| + 3|G_{ij}| = 10$ for all $j \in I \backslash \{i, -i\}$, consequently, $|G_{ij}| \leq 3$. As $|I \backslash \{i, -i\}| = 12$, we conclude that $|G_{ij}|=3$ for all $j \in I \backslash \{i, -i\}$.

We note that $|G_{ijk}| \leq 1$ for any $j, k \in I \backslash \{i, -i\}$ and $|j| \neq |k|$, otherwise, there will be two distinct codewords in G_{ijk} covering a same word of type $[\pm 1^3]$.

Let $W_1 \in G_i$ such that $W_1 \in G_{i\alpha\beta\gamma\delta}$, where $\alpha, \beta, \gamma, \delta \in I \backslash \{i, -i\}$ and $|\alpha|, |\beta|, |\gamma|, |\delta|$ are pairwise distinct. Then, $|G_{i\alpha}| = |G_{i\beta}| = |G_{i\gamma}| = |G_{i\delta}| = 3$ and the codewords of G_i satisfy the conditions presented on Table 1. That is, for any $W \in G_i \backslash \{W_1\}$ there is a unique element $\theta \in \{\alpha, \beta, \gamma, \delta\}$ such that $W \in G_{i\theta}$.

Let $J = \{\alpha, \beta, \gamma, \delta\}$ and $J^- = \{-\alpha, -\beta, -\gamma, -\delta\}$. Consider $K = I \backslash (\{i, -i\} \cup J \cup J^-)$, that is, $K = \{p, -p, q, -q\}$.

Let W, $W' \in G_{i\theta} \backslash \{W_1\}$, where $\theta \in J$, such that, $W \in G_{i,\theta,j1,j2,j3}$ and $W' \in G_{i,\theta,j4,j5,j6}$. We must impose $j_1, j_2, \ldots, j_6 \in (J^- \cup K) \backslash \{-\theta\}$ and distinct between them.

Note that, there are no $U \in G_{ixyz}$ with $x, y, z \in K$, since $|x|$, $|y|$ and $|z|$ have to be distinct.

As $|\{j_1, j_2, \ldots, j_6\} \cap J^-| \leq 3$, the codewords of G_i satisfy the conditions on Table 2, where $x_s, y_s, z_s \in K$ are distinct for all $s \in \{1, 2, 3, 4\}$. Considering the codewords W_2, W_4, W_6 and W_8, are exhausted all possible combinations between the elements of K. Thus, suppose, without loss of generality, that $x_1 = p$, $y_1 = q$ and $z_1 = -p$. Consider, without loss of generality, that $x_2 = -p$ and $y_2 = q$, see Table 3. Then, $u_1, u_2, u_3, u_4 \in J^-$ and are distinct between them, otherwise there are two distinct codewords in $\{W_2, W_3, W_4\}$ covering a same word of type $[\pm 1^3]$. As $|G_{ik}|=3$ for all $k \in K$, there exists $z \in \{z_3, z_4\}$ such that $z = q$. However, this is not possible since we must impose $W_7 \in G_{i,\gamma,q,u2,u3}$ or

$W_9 \in G_{i,\delta,q,u2,u3}$, that is, W_7 and W_3, or, W_9 and W_3 cover a same word of type $[\pm 1^3]$, contradicting the definition of perfect Lee code. □

From the results above we get a range of possible values for g, see next corollary.

Corollary 11. For $g = |G|$, $9 \leq g \leq 22$.
Proof. From Lemma 9 we have $3 \leq |G_i| \leq 9$ and by Lemma 10 we know that $|G_i| \neq 9$, thus $3 \leq |G_i| \leq 8$ for all $i \in$ I. As $g = |G|$ is a nonnegative integer number satisfying $g = \frac{1}{5}\sum_{i \in I} | G_i |$, and |I |=14, we get $9 \leq g \leq 22$. □

We believe that to prove the impossibility of $|G_i| = x$ for x a positive integer number satisfying $3 \leq x \leq 8$ we will need to recur to another sets of T, in particular subsets of F, since the information from codewords of G_i will not be enough to prove the requested. Thus, we expect that the proofs of these cases will be more difficult when compared with the proof of the last lemma.

4. Conclusion

The Golomb-Welch conjecture states that there are no $PL(n, r)$ codes for $n > 2$ and $r > 1$. In this paper we deal with $PL(n, 2)$ codes which are, in the opinion of some authors, the most difficult cases of the conjecture. Since, it is proved the nonexistence of $PL(n, 2)$ codes for $3 \leq n \leq 6$, we are interested on proving, particularly, the nonexistence of $PL(7, 2)$ codes. We introduce a possible strategy, influenced by Horak method [6], to prove the nonexistence of such codes. By contradiction, it is assumed the existence of a $PL(n, 2)$ code $M \subset Z^n$, such that, $O \in M$. As all words $W \in Z^n$ such that $\rho_L = (W, O) \leq 2$ are covered by the codeword O, our aim is to prove that it is not possible to cover all words $V \in Z^n$ satisfying $\rho_L(V, O) = 3$ without superposition between codewords of M. Considering, in particular, $PL(7, 2)$ codes, Horak have proved [6] that the cardinality of the sets composed by codewords which cover all these words V satisfy the following system of equations,

$$\begin{cases} a + b + c + d = 14 \\ b + 2c + 2d + 4e + 3f = 168 \\ d + e + 4f + 10g = 280. \end{cases}$$

We are interested in finding solutions for these conditions which satisfy the definition of $PL(7, 2)$ codes. In this paper we show that the variability of the parameter g must be between 9 and 22. In future work we hope to give a lower range of values for g and, consequently, to prove that any solution for this system contradicts the definition of perfect Lee codes.

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Tables. Codewords of G_i

Table 1.

W_1	i	α	β	γ	δ
W_2	i	α			
W_3	i	α			
W_4	i	β			
W_5	i	β			
W_6	i	γ			
W_7	i	γ			
W_8	i	δ			
W_9	i	δ			

Table 2.

W_1	i	α	β	γ	δ
W_2	i	α	x_1	y_1	
W_3	i	α	z_1		
W_4	i	β	x_2	y_2	
W_5	i	β	z_2		
W_6	i	γ	x_3	y_3	
W_7	i	γ	z_3		
W_8	i	δ	x_4	y_4	
W_9	i	δ	z_4		

Table 3.

W_1	i	α	β	γ	δ
W_2	i	α	p	q	u_1
W_3	i	α	$-p$	u_2	u_3
W_4	i	β	$-p$	q	u_4
W_5	i	β	z_2		
W_6	i	γ	x_3	y_3	
W_7	i	γ	z_3		
W_8	i	δ	x_4	y_4	
W_9	i	δ	z_4		