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Linear Algebra in New Environments (LINE) – An Assessment

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Abstract

We report on an analysis of student survey results based on their experiences using LINE (LInear Algebra in New Environments) techniques and curriculum materials teaching linear algebra. Our goals were to (a) to create a professional learning community across STEM disciplines, (b) to combine expertise in content and pedagogy in designing effective instructional practice, and (c) to use learning theories to support the conceptual alignment of content and pedagogical goals. In particular, our approach combines the use of domain-specific problems, APOS learning theory, and the development of the professional learning community. This set of practices was developed and deployed across four diverse institutions, in diverse linear algebra course, and was effective across this diversity. We discuss the development of teaching materials and present reflections from students which were shared after these "modules" were used in classes. These show that the collaboration between mathematicians and mathematics educators has been extremely valuable in our rethinking of instruction; student reflections point to motivation and deeper conceptual understanding. This paper is part of a series of papers published about this work.

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Introduction

This paper reports on a subset of findings from ongoing research projects that have been partially supported by the National Science Foundation (NSF DUE 0442574, 0837050). The projects were designed to incorporate learning theories into the design and teaching of undergraduate linear algebra. . Course materials, including a variety of modules that involve student explorations of applications of linear algebra, were developed to reflect the APOS framework for concept development as individuals learn mathematics (Asiala, Brown, DeVries, Dubinsky, Mathews, and Thomas, 1996). Detailed information about the design and implementation of the projects can be found in other publications (Martin, Loch, Cooley, Dexter, and Vidakovic, 2010; Cooley, Vidakovic, Martin, Dexter, Loch, and Suzuki, submitted). This paper focuses on the analysis of written reflections of participating students on their learning and experiences in LINE project linear algebra courses taught at three institutions from 2011-2013.

The Linear algebra In New Environments (LINE) project was designed to promote a reflective, collaborative culture of teaching and learning among STEM discipline faculty. The project integrates (a) the study of important mathematical content, (b) the use of applications, and (c) reflection on mathematical learning theories by faculty and students in the context of an advanced undergraduate linear algebra course. The development and implementation of this course involved co-teaching and collaboration among faculty with expertise in a variety of areas, including mathematics, computer science and mathematics education. The LINE model includes professional development of college faculty who will form professional learning communities to support implementation of innovative teaching modes.

The project has three main goals:

- Motivate students by connecting mathematics to their experiences;
- Match the analysis of content to theories of learning and instructional models; and
- Provide collaborative support for faculty so that expertise in content and pedagogy are used to design effective instructional practice.

The most novel characteristic of this project is the integration, within a single course of the study, of content and applications, in a manner infused by current theories of teaching and learning mathematics at the undergraduate level. Furthermore, this project was only possible because of the novel cooperative efforts of faculty with expertise in each of the three areas: content, applications, and educational theory.

This project was based on previous work completed by the PIs in collegiate mathematics learning theory, as well as in linear algebra. The previous successes from these projects were brought to bear to this project, thus building on what we know can work. There is a strong interest in mathematics, and particularly in linear algebra, because of its importance for secondary mathematics, computer science, abstract mathematics and engineering. This project seeks to transform STEM education at the participating institutions by demonstrating the impact of curriculum design based firmly in a rich collaboration among disciplines that contribute content, application, and educational theory. We are working to stimulate a genuine dialogue about how STEM education could evolve, and the way that mathematics professors and others think about how their students learn mathematics.

2. Four primary goals for student learning

While LINE projects seek to collaboratively promote change in undergraduate course materials and instruction through an interchange of ideas and reflection by mathematicians and educators, the ultimate goal of the projects is to strengthen student learning of undergraduate mathematics, specifically in linear algebra. We identified four goals for student learning that we hope to promote through our work with the project team and course instructors.

1. Students will gain more conceptual understanding of mathematical content.
2. Students will be more actively engaged in their own learning.
3. Students will gain self-confidence in their capacity to do mathematics.
4. Students will develop a broader impression of the nature of mathematics.

While the content area was linear algebra, chosen because of its significance as a core course for both mathematics majors and many other disciplines and for its rich diversity of applications, the design ideas and goals could easily be adapted to other mathematical domains. Since the participating institutions ranged from two-year colleges to research intensive universities, the course materials were developed in a wide range of different settings. A sample module is provided in Appendix A.

3. Methodology

As mentioned earlier, this paper reports on the specific aspect of soliciting reflections from participating LINE students. We have collected a variety of data in terms of assessment. To investigate the impact of these courses we asked instructors and students questions about their experiences both during and after the conclusion of the courses. We analyzed written student work and conducted problem solving group and individual interviews. The focus of this paper is on written student reflections about their experiences and learning in the courses to determine the extent to which their perceptions matched those of our stated goals for student learning.

Students at three universities – Brooklyn College - City University of New York, Georgia State University, and North Dakota State University – responded to open-ended survey questions related to their learning and experiences in the courses. The survey questions were:

1. In linear algebra, we worked with matrices throughout. In addition to working with, and the deep use of, matrices, what would you say are the three most significant ideas that you learned in this class? Why do you think so? Please explain clearly. (If you want to put four, it's okay.)
2. We know that linear algebra is a topic in itself and you may not have studied this subject before. So, other than the actual linear algebra content, was this mathematics class different than other math classes you have taken? If so, in what ways? Please explain clearly.
3. Other than the applications contained in them, in what ways were the modules (quizzes) helpful as a learning tool for you personally? If you think they were not helpful, please explain why.
4. What do you believe was the most striking aspect of the course that will stick with you after the class is over? Please elaborate with a few sentences.
5. In what way did working with a partner help or hinder your learning? What was the most useful part of the module? Would you like to have more assignments in module form with a partner?

These data were compiled for each question. The PIs then worked collaboratively to review the student comments carefully and to extract common themes.

Using these initial themes, we again reviewed all of the comments and classified each response as incorporating one or more of those themes. During this initial classification phase, we had extensive discussions that led to clarifications, expansions and revisions of the original themes. We also found that a large proportion of comments could not be classified with the existing themes. We then as a research team made a third pass through all of the comments, classifying each with our agreed upon final theme codes or as un-codable.

An important aspect of the coding is that we needed to describe the students' perspectives on their learning and experiences. Therefore, the codes reflect student perceptions. The titles that we used for our codes are an interpretation of the student perceptions. They are not meant to be formal definitions of the terms in any way. It is a language that we developed collaboratively through the extensive process of reviewing, discussing, sifting, and analyzing student reflections.

The final codes we used are given below along with their descriptions:

C: Abstraction. Students identified abstraction in linear algebra, discussed conceptual understanding, recognized recurring mathematical ideas across multiple contexts and/or explicitly mentioned the theoretical nature of what they had learned.

N: Nature of mathematics. Students discussed how their perceptions of mathematics as a whole have developed in the class. They may have indicated new insights into a broader interpretation of mathematics or discussed how what they have learned fits in with their prior knowledge.

M: Metacognition. Students reflected on their own understandings, especially how they have changed or been influenced by the class.

L: Language. Students specifically discussed, and perhaps confessed to some some confusion with, language, vocabulary or terminology.

A: Applications. Students discussed the value *to their learning* of applications or modeling real life situations.

G: Group work. Students discussed benefits of collaborating with other students, such as talking over ideas, solving problems jointly, or the interactive dynamics.

R: Responsibility for learning. Students discussed their agency in their own learning. They may have mentioned the needs for stronger study skills, new habits of thought, or increased self-motivation.

T: Traditional learning styles. Students indicated a preference for traditional models of teaching learning. They may have identified difficulties with working with others, problem solving in the modules, or a preference for a textbook, teacher-centered, or lecture type of classroom experience.

P: Procedural. Students emphasized the computational, mechanical, algorithmic aspects of linear algebra.

Results

Some student reflections were more extensive than others and, therefore, touched upon more than one theme. While most comments were coded with just a single code, some were coded with two or three codes. Table 1 below summarizes the distribution of codes.

Table 1. *Frequency of Codes*

Code	Single Frequency	Combined Single and Multiple Frequency	**Dual Code**	Dual Frequency (appeared together)	**Triple Code**	Frequency
A	19	32	**AC**	6	**ACM**	2
C	21	38	**AM**	2	**AMT**	1
G	18	20	**AR**	2	**CMR**	1
L	6	10	**CL**	1	**CNR**	1
M	13	35	**CM**	3	**LMR**	1
N	9	13	**CN**	2		
P	9	9	**CR**	1		
R	12	24	**GM**	2		
T	6	11	**GT**	1		
subtotal	**113**	**192**	**LR**	1		
			LM	1		
Uncoded	119	119	**MN**	1		
			MR	5		
			MT	1		
Total		**311**		**29**		**6**

We related these codes to our four goals for student learning as follows:

1. Students will gain more conceptual understanding of mathematical content. *Applications*(32) and *Abstraction*(38).
2. Students will be more actively engaged in their own learning. *Applications*(32), *Group work*(20), *Metacognition*(35), and *Responsibility for learning*(24).
3. Students will gain self confidence in their capacity to do mathematics. *Metacognition*(35) and *Responsibility for learning*(24).
4. Students will develop a broader impression of the nature of mathematics. *Nature of mathematics*(13), *Language*(10), and *Procedural*(9).

Goal 1: Students will gain more conceptual understanding of mathematical content. A (applications) and C (conceptual nature of the material, the most frequent comment) were two of the three most commonly mentioned ideas by students about the course. Over a third of the coded responses, 70 out of 192 (36%), indicated that applications somehow improved their learning or that the course deepened their conceptual understanding. They appeared together as a double code, in other words, student responses included indications of both ideas, more than any other pair (8 times out of the total of 35 multiple coded items). Below is a sample of a response that was coded both A and C:

A, C – 'As much as I think some people had problems with the modules (they take up time that could be spent being "more productive"), the modules are definitely beneficial for the class since they show a real world application of the ideas we're learning. Surprisingly, that is something so foreign to many classes. We usually learn concepts to apply them to constructed textbook problems. It is refreshing to see your work conquer a real problem, and it gives you a greater sense of usability and relevance to the concepts you are learning.'

A sample coded just A:
A – 'The problem with academia is that we learn a lot, but most of the time knowledge seems sterile and not applicable in real-life. It makes a course so much more interesting, if we know where we can use all the new ideas that we are learning. Undergraduate students do not have any significant work experience. It's only after years as professionals that they understand what they were learning is useful!'
Goal 2: Students will be more actively engaged in their own learning. For the second learning goal, we mapped the codes M (metacognition), G (group work), A (applications) and R (responsibility for learning). Of the 192 codes awarded, 121 or 63% were in these categories. The survey questions seemed to parse out different aspects of the engagement in learning and most of the responses were labeled with single codes. M (35 occurrences) was the second highest code used and relates to Goals 2 and 3. M was one of the codes indicating that students described being more actively engaged. Code A (applications helped learning) was the third most frequent, indicating the perceived value to their learning of modules that applied linear algebra concepts. Twenty student comments were about positive effects on their learning via group work during the course. Below are samples of student reflections that used the codes M, A, G or R:
R, A - 'I discovered that I am still interested in math but that I do not invest the time I should into the mathematics as I should. I also did see what I was doing with math. And learning on how I can apply math helps & motivates me to better understanding math. Talking!'
G, M – 'Even though working with a partner or partners slow my completion time, I would like to have more assignments in module form with a partner/partners. Since they count for a small portion of the grade, its really more about learning the ideas. Working with a partner will slow myself down and make sure I really understand what's going on, because I learned that if you can't teach it to other people, you probably don't really know what you're talking about. Also I'm all for having students communicate with each other.'
Goal 3: Students will gain self-confidence in their capacity to do mathematics. Students discussed their confidence in being able to solve problems as a result of the module experience in various ways. Persistence and success after persistence were a part of the theme for these comments. 59 coded comments, 52%, related to this goal. These included the two codes for Metacognition and Responsibility:
M, R – 'I learned so much, I would say the approach to proofs is where I learned the most I developed a new way of think which was much more deductive and solid compared to what I had used in most of my other courses. I learned that in order to really understand a concept you have to think about different approaches, and maybe do some research on the side.'
M – 'Being able to write down what I already know in words and describe my thought process which is a lot harder than it seems.'
M – 'Linear Transformations, seems at first a very simple concept. However the deeper we got into the study of them the more careful I had to be. I

wouldn't say I don't understand linear transforms, but that particular questions confuse me at times. The full importance of this kind of concept is hard to talk about, however linear transforms are required in many types of applications and studies from programming to problem solving.'
R – 'That even though the material may seem different and difficult at first, that if I stick with it, it won't be as difficult as it seemed to be.'
R – 'I discovered that if I really sit and think about a problem, I will eventually come to an answer. Obviously this will help me in the future, and has increased my motivation to really work at a problem.'
Goal 4: **Students will develop a broader impression of the nature of mathematics.** Twenty-three coded comments (12%) were determined to be about students' broader views of mathematics. Students mentioned that their views of the nature of mathematics had been broadened in some way by the class or they discussed the implications of language in mathematics based on their experiences in this class:
L, R – 'The aspect of explaining everything that you do, mathematical vocabulary needed as well as the need to always be ahead and prepared.'
N – 'Also, the ideas behind what is done with the matrices probably pops up throughout math classes all the time, and they will most likely prove useful.'

Discussion

Student reflections strongly supported the impression that three of the four project goals for student learning had promoted the desired gains. For the first three learning objectives, students made a large number of comments that this experience helped them to grow in these ways. This has several implications for pedagogy. Firstly, although many professors are concerned that introducing such projects may take away from the time needed to cover topics, student responses indicate that, to the contrary, they believed their understanding had been deepened by engaging in these problem-solving situations. At one site, the professor even introduced ideas via modules prior to class. Students in the class commented that this was often helpful for them in learning the material and that they felt better prepared by doing so.
There were some students who voiced a preference for a more traditional approach to instruction, something more teacher or book centered or strictly with lectures rather than working collaboratively with their peers. However, these comments were definitely in the minority. Only 11 responses, 9%, were coded T for a preference for traditional classroom experience:
T – 'I felt the homework assignments and the book gave me a better understanding of the material.'
T – 'Different: The biggest difference between this class and others was the presentation component. When I took the class I didn't mind, but I don't think it's the best way to run a math class. While being able to effectively communicate your ideas is important, it can also be really intimidating to some people. Another thing I notices was that most of the assignments had one or

two problems that were an application of the concepts instead of just purely abstract. I did enjoy having these to give a clear example of how to make use of the theorems and ideas.'

Finally, in terms of walking away with an impression that this course was about concepts and ideas rather than procedures and algorithms, only 9 students said that procedures were a focus of linear algebra. We believe that this is significant in that the problem solving approach was designed to pull students away from this perspective and we appear to have been quite successful in doing so. Here is an example of a response coded P:

P – 'I think the three most important ideas learned in class are row reducing matrices, multiplying matrices III and transformations. Row reducing a matrix really helps you understand if the "vectors" contained in the matrix are linearly dependent or independent. Transformations were important as well since they cover how multiplying a matrix with a vector changes the vector, and leads to eigenvalues/eigenvectors. Multiplying matrices is just fun to be honest.'

Conclusions and Relation to Other LINE Papers

This paper is part of a series of research reports about LINE outcomes. These include papers on the learning of linear algebra from an APOS perspective (Meagher, Cooley, Martin, Vidakovic and Loch, 2006), coordinating learning theories with linear algebra (Cooley, Martin, Vidakovic and Loch, 2007), the effect of the simultaneous study of linear algebra and learning theories (Vidakovic, Cooley, Martin and Meagher, 2008), integrating learning theories in applications for linear algebra (Martin, Loch, Cooley, Dexter and Vidakovic, 2009), and, finally, a description of the development of modules, with samples (Cooley, Vidakovic, Martin, Loch, Suzuki, and Dexter, in print; Cooley, Vidakovic, Martin, Dexter, Loch, and Suzuki, submitted).

This paper was in reference to students' perceptions of the course and of their own learning. Our next goal is to examine students' mathematical efforts. We have gathered their written work in the modules and videotaped groups of students while they solved problems jointly. Finally, we interviewed a subset of these students to further probe their thinking. We hope to determine if the students' perceptions of their understanding are demonstrated via their descriptions of their understanding.

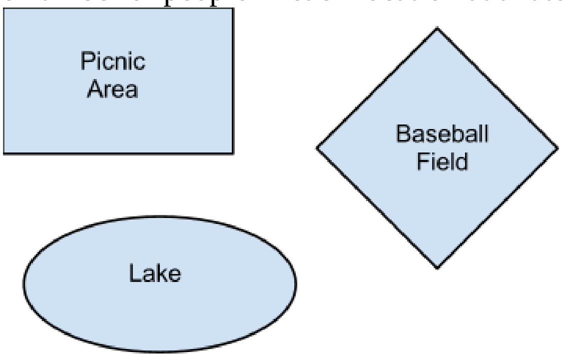
Appendix A – Sample Module

Module 2 - Traffic Models

Complete this module with your partner. Write out all of your answers in complete sentences on a separate sheet of paper or NO credit will be given. Hand in your answers with both names on it.

Introduction

Suppose you have a park with three areas: the picnic area; the lake; and a ballpark. If you know how many people are in each location at a given time, can you predict the number of people in each location at a later time?



Assumptions

In order to model the number of people at each location, we'll have to make some assumptions on how the people move. For now, we'll assume time is measured in hours, and make the following assumptions:

- Of those in the picnic area at time t, 70% will be at the baseball field one hour later; 20% will be at the lake one hour later; and the remaining 100 - 70 - 20 = 10% will still be at the picnic area one hour later.
- Of those in the baseball field at time t, 20% will be at the picnic area and 40% will be at the lake one hour later, with the remainder staying at the baseball field.
- Of those at the lake at time t, 30% will be at the baseball field one hour later and 50% will be at the picnic area one hour later, with the remainder staying at the lake.

Questions – Important! Explain your answers in each question with full sentences or NO credit will be given:

1. Suppose that, at noon, there are 100 persons in each area. How many will be in each area one hour later? Two hours later?
2. Suppose there are x persons at the picnic area, y persons at the baseball field, and z persons at the lake. How many persons will be in each of the three areas one hour later? Express the total as a sum, using x, y and z.
3. Suppose that, at noon, there are 100 persons in each area. How many persons were in each area one hour *before* noon? (It's okay to get a negative value here.)
4. Is it possible for the number of people in each location to be the *same* at time t and time t+1? If this is not possible, explain why; otherwise, determine how many people would be in each area.

5. Express the number of people in each area at time t + 1 as a linear transformation of the number of people in each area at time t. In other words: Suppose the number of people in each area at time t is (x, y, z). What linear transformation A(x, y, z) will give you the number of people in each area at time t + 1?
6. How would you change the model to reflect that people both leave and enter the park?

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