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Using Analogies in Geometrical Lessons

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Abstract

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Thinking in analogous structures leads through all areas of daily life and in scientifical thinking, too. Analogies are often used vaguely, there are ambiguous and incomplete or incomplete cleared analogies but analogy can also reach a high level of mathematical precision. Corresponding analogies are used in mathematics in different didactic situations.

using analogies while operating:

analogy when exceeding tenner limits: **3 + 5 = 8** ☛ 1**3** + **5** = 1**8**

using analogies while creating terms:

geometric terms as sets of points: circle **k** and sphere Φ

$$\left\{ P_k \in R^2 \middle| \bigvee_{n=1} M_k : \quad d(P_k, M_k) = \text{const.} \right\}$$

$$\left\{ P_\Phi \in R^3 \middle| \bigvee_{k=1} M_\Phi : \quad d(P_\Phi, M_\Phi) = \text{const.} \right\}$$

using analogies while finding theorems:

The area of a quadrilateral ABCD of chords of a circle arises from half the size of its perimeter and the lengths of the sides a, b, c and d as well as this

$$\mathbf{A} = \sqrt{(\mathbf{s} - \mathbf{a})(\mathbf{s} - \mathbf{b})(\mathbf{s} - \mathbf{c})(\mathbf{s} - \mathbf{d})}$$

(BRAHMAGUPTA'S formula)

The area of a triangle (a three-sided chord figure) ABC arises from half the size of its perimeter and the lengths of the sides a, b and c as well as this

$$\mathbf{A} = \sqrt{(\mathbf{s} - \mathbf{a})(\mathbf{s} - \mathbf{b})(\mathbf{s} - \mathbf{c})\mathbf{s}}$$

(HERO'S formula)

with $s = \frac{1}{2}(a + b + c)$

using analogies while proving ***(figure 1)***

using analogies **while solving tasks:** *a given* ***rectangle*** *has to be changed in a square* *<u>equal in area</u>.*

As possible tasks which activate for thinking in analogous structures the two following ones would be conceivable:

- *A* ***rectangle*** *given by the lenghts of its sides has to be changed into a* ***square*** *<u>equal in perimeter.</u>*
- *A* ***cuboid*** *given by the lenghts of its sides has to be changed into a* ***cube*** *<u>equal in volume</u>.*

Both tasks are different by the choice of the reference systems. While in the first case the flat reference system will not be left, for the solution of the second formulated task one has to change from the flat reference system to a spatial reference system. For the analysis of the analogical conclusions possible

solution ways are be outlined first in **figure 2 and 3** and the considerations analogous to the last given task treated before can be gathered from the illustration in **figure 3**.
At closer analysis of the considerations to be performed in the solution process the nature of the analogies is shown by comparing correspondences when transferring in the spatial reference system, for example:

rectangle - upper side	|	*rest rectangle - area*
cuboid - upper face	|	*rest column - volume*

half rest rectangle - opening out corners
half rest column - opening out edges

In the circle of modelling real *situation → mathematical model → mathematical solution → validating the found solution* an important difference is existing between the steps to find a mathematical model and to work in the model. The last one is dominated by working with mathematical routines, the other especially by heuristical working. Pupils could be successful in the complete process of modelling only if they are successful in the phase of creating a mathematical model using heuristical methods. One method is the principle of using analogies.

Using analogies proves therefore as a *replacing* in a certain way. Going beyond Thomas of *AQUIN* who saw an analogy as a correspondence between relations, *POLYA* formulated still more clearly:

"Analogy has something to do with 'similarity' but this concept (according to the geometry) is more narrow than the concept analogy. Similar things agree on any regard, analogous things agree on certain relations between their corresponding parts.“ /6/, p. 52

For using analogies you essentially distinguish between two phases:

- Forming the analogy, comparing of objects and finding of correspondences, transfering and formulating the assumption found by analogy.
- The proof of the validity of the found by analogy statement, if necessary under inclusion of broader analogy considerations.

Not in every case the process must be passed through completely. The proof can be renounced from time to time since this likes to seem unmotivated to the pupil from the didactic point of view because of the low heuristic room to move. Therefore the case of the distance definition in the Euclidean plane ($\mathbb{R}^2$) and the Euclidian space ($\mathbb{R}^3$) is an excellent example. This renunciation can also be justified out of didactic considerations. Reducing mathematical treatises to the justification of the won knowledge on a minimum in a finding process might be sensible. An example of this is given by **figure 4** in a textbook of mathematics (/5/, S. 168):
A prismatic or cylindrical vessel has a base area of 12 cm^2. The vessel shall be filled with water up to a limit of 4.5 cm. How much cubic centimeters of

water does the vessel then contain?
We fill water into the vessel up to a limit of 1 cm once and wonder that exactly one cubic centimeter of water then is above every square centimeter of the base ***independent from the form of the base.***
The first layer of 1 cm of height contains therefore exactly as many cubic centimeters of water as the base contains square centimeters. In our vessel is water at a depth of the layer of water of 1 cm over the base of 12 cm^3.
If we fill the vessel with water up to a limit of 4.5 cm now we will have 4.5 cm^3 of water above every square centimeter of the base. To calculate the volume (in cm^3) well, one must multiply the base area (in cm^2) by the altitude (in cm).

$$V = 12\,\text{cm}^2 \cdot 4{,}5\,\text{cm} = 54\,\text{cm}^3$$

***general**:* $\quad V = A_G \cdot h$

The second part of an analogy process, the phase of the proof of the validity, is not less important. It may include aspects which are equally essential for the development of thinking. On the one hand it may be to give reasons for the soundness of the used analogy if a complete mathematical proof is not possible with pupils. On the other hand it could be the aim to rob a reported analogy well of its reliability as the following example shall indicate.
On the search for the area formula for triangles a triangle, characterized by a side and the accompanying altitude of the same length, is chosen. The area formula for triangles can be found by the changing of the triangle into a rectangle equal in area (logically two squares here).

$$A = \frac{a}{2} \cdot a \quad \Rightarrow \quad A = \frac{g}{2} \cdot h_g$$

Since no fault is made on the way so far because of the choice of the special case, the similar spatial analogon, in this case the cube, allows the following conclusions to find a formula for calculation the volume of a pyramid using **figure 5**.

$$6 \cdot V_{Pyr} = V_W = a^3$$

$$V_{Pyr} = \frac{1}{6} \cdot a^3 = \frac{1}{6} \cdot a^2 \cdot a$$

$$V_{Pyr} = \frac{1}{6} \cdot A_g \cdot h$$

Although the made considerations seem to be plausible for the so far untested analogy to some extent, they are not correctly in the result anyway.
The use of analogies is subordinated to the respective didactic situation in the lesson. However, also certain types making analogies can be distinguished in general.

Reducing or extending the number of defining parts which determines a mathematical object for (number of edges, …)

With the treatment of the kinds of quadrilaterals in mathematics this analogy principle can be used depending on the intentions of the lesson more or less clear. The formation of the concepts already shows this. That will be particularly clear at concept formations by defining strictly with the use of major and minor terms.

trapezium	A **trapezium** is a (convex) **quadrilateral** with a couple of parallel opposite sides.
parallelo-gram	A **parallelogram** is a (convex) **quadrilateral** with two couples of parallel opposite sides.
rectangle	A **rectangle** is a (convex) **quadrilateral** with two couples of parallel opposite sides and a right angle.
square	A **square** is a (convex) **quadrilateral** with two couples of parallel and equally long opposite sides as well as a right angle

Apart from that the used analogy could also be clarified by systematizing considerations to the found area formula which the following chart shows.

trapezium

$$A = \frac{a+c}{2} \cdot h$$

parallelogram

$$A = \frac{a+c}{2} \cdot h = \frac{a+a}{2} \cdot h = a \cdot h$$

rectangle $A = \frac{a+c}{2} \cdot h = \frac{a+a}{2} \cdot b = a \cdot b$

square

$$A = \frac{a+c}{2} \cdot h = \frac{a+a}{2} \cdot a = a^2$$

The used analogy has an effect only indirectly. It is better to see in tasks to change the number of defining parts of a mathematical object. In far interpretation of this thought tasks which are based on this can be understood to adapt definitions to a concept for which they originally were not thought.

Triangles wi th the side lengths a. b, and c which follows the formula $a^2 + b^2 = c^2$ are named pythagorean triangles.

Possible statements as

Quadrilaterals with the side lengths a, b, c and d are named pythagorean quadrilaterals if they follows the formula

(1) $a^2 + b^2 = c^2 + d^2$ (2) $a^2 + c^2 = b^2 + d^2$ (3) $a^2 + b^2 + c^2 = d^2$

allow considerations whether then such quadrilaterals exist at all. The second aspect of an analogical conclusion is taken into account particularly adequately to legitimize the correctness of the performed end reasonably or to expose the found result by using an analogy like in the example (3) with that – at a lower level, of course, since, mostly, the proof of the existence of one of those defined quadrilateral suffices.

to (1): M shall be the center of one diagonal of a quadrilateral ABCD (e.g. the diagonal AC), the quadrilateral is exactly then a pythagorean figure in the described meaning, if the vertices B and D lie on a circle around M with an arbitrarily radius.

to (2): In this case not neighboring but opposite four sides are "coupled squarely". The considerations would lead the statement that a quadrilateral ABCD is exactly then pythagorean if the diagonals stand vertically on each other. You can speak of generalized kites in a certain way.

The statement of the pythagorean theorem can also lead to further-reaching questions which take in addition geometric analogies between the $\mathbb{R}^2$ and the $\mathbb{R}^3$ into account whose variety is reflected in the analysis of the following analogical conclusions (compare **table 1**):

These considerations lead to the following:

case II : $d^2 = \dfrac{d_1^2 + d_2^2 + d_3^2}{2}$ case III: $d^2 = a^2 + b^2 + c^2$.

A further-reaching analogy would be the case of a perpendicular tetrahedral vertex:

At the cut of a cube along two area diagonals colliding in a vertex of the cube a tetrahedron arises with a perpendicular tetrahedral vertex

On the one hand in addition to present contents calculations at the cube but also in analogy to the connections between different diagonals at the cube examined before the considerations are extended now to the tetrahedron with the following generalized question

For a simple right-angled corner O in a tetrahedron the areas A, B and C of the three in O colliding faces are given; one shall determine the area of the face D opposite O.

Pupils could guess that for the areas mentioned in the task $D^3 = A^3 + B^3 + C^3$ must be valid, because for the space diagonal in a cube is valid $d^2= a^2 + b^2 + c^2$ (or in a rectangled triangle for the sides lengths is $c^2 = a^2 + b^2$). This analogical conclusion even seems to be sensible at the transition of $\mathbb{R}^2$ (square) to the $\mathbb{R}^3$ (cube).

The solution of the task then leads to a surprising result (applies to the dimensions of the mentioned areas is $D^2 = A^2 + B^2 + C^2$) and underlines the necessity of the check of an apperantly convincing analogical conclusion.

Similar correspondences are also found between simple flat and spatial things.

inscribed circles and escribed circles or spheres

circle (at the triangle) - **spheres** (at the tetrahedron)

circumcircle - circumsphere
inscribed circle - inscribed sphere
escribed circle - escribed sphere

Some knowledge is arranged exemplarily in the following chart. All considerations require or transport an adequately distinctive spatial imagination.

triangle		tetrahedron	
circumcircle	contains the three (not collinearly situated) vertices of the triangle; the center lies on the common cut of the average perpendiculars	circum-sphere	contains the four (not coplaned situated) vertices of the tetrahedron; the center lies on the common cut of all average perpendicular levels.
inscribed circle	touch the three sides of the triangle tangentially from inside; the center lies on the common cut of the bisector of the interior angle	inscribed sphere	touch the four sides of the tetrahedron tangentially from inside; the center lies on the common cut of all bisector planes of the six edge angles
escribed circle	touch the three sides or their pro-longations of the triangle tangen-tially of the outside; the center lies on the common cut of a bisector of an interior angle with the bisectors of the two outer angles	escribed sphere	touch the four limiting areas of the tetrahedron or their expansions tangentially of the outside; the center lies at the "base" of the tetrahedron on the common cut of the bisector planes of the outside edge angles

Analogies can finally lead constructive geometry (in the sense of spatial contemplation, imagine and thinking) to its limits and can also open the entry to the $\mathbb{R}^n$ for instance by gradual construction of the hypercube. The aim is to bring the constructive fantasy of the pupils (in the sense to the aim of learning *spatial contemplation*) to their scientific limits within the school mathematics subject by analogies, aspects which will define spatial contemplation when fortune demanded, raised and excercised there in (a) higher dimension(s).

A first formal notation arises from the following considerations **(figure 6)**:

If a point moves from O along a unit vector $\vec{e}_1$ then the unit of length arises with its end-points (0|0) and (0|1). The unit of length itself with the traces of its end-points moves along for the unit vector $\vec{e}_2$; the unit square arises with the vertices (0|0), (0|1), (1|0) or (1|1),.Finally at further moving of the unit square in direction of $\vec{e}_3$ the unity cube arises. The independence of the result from the order of the directions is easily understandable. A fourth moving determined by the independent vector $\vec{e}_4$ with respect to start and ending position of the cube as well as the traces of its eight vertices is purely deductively reasonable. The result would be the hypercube with the length 1 of all of its edges. **Table 2** shows further possible anologies.

Transposing assumption and assertion of mathematical theorems, formation other combinations

The dual theorem at the *DESARGUES'* theroem is his own reversal. Examples from the lesson with general educational value based on this type of an analogical conclusion are numerous. The variety of possible starting points can be seen at the analysis of the group of the theorems of proportional segments. Only two of the three formally possible reversals which have to be checked for their truth value are respectively represented for two theorems of proportional segments in **figure 7**.

The examples described till now have referred to local contents of mathematics mostly. The analogies are mostly based on the analogy between the $\mathbb{R}^2$ and the $\mathbb{R}^3$. This can rightly therefore be understood as a specific strategy according to a fundamental leading idea of mathematics.

For this many further examples could be given, absolutely every one is worth of attention for mathematics:

- circle around a point - circle around a line
- axiomatic procedure for defining the absolute value of a vector - axiomatic procedure for defining the dotproduct

$$\left|\vec{a}\right| > 0 \quad \forall \vec{a} \neq \mathbf{o} \qquad \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a} \qquad \vec{a}\cdot\left(\vec{b}+\vec{c}\right) = \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$$

$$\left|\vec{a}\right| = \left|\overrightarrow{-a}\right| \quad - \quad \left(k\cdot\vec{a}\right)\cdot\vec{b} = k\cdot\left(\vec{b}\cdot\vec{a}\right) \qquad \vec{a}\cdot\vec{a} > 0 \quad \forall \vec{a} \neq 0$$

- description of a line in the P^2 without a parameter - description of a line in the P^3 without parameters

$$\bigwedge_{i=1}^{\infty} P_i \in g : \mathbf{n}_g \cdot \mathbf{a}_g = \mathbf{n}_g \cdot \overrightarrow{P_0P_i} = 0 \qquad \bigwedge_{i=1}^{\infty} P_i \in g : \mathbf{a}_g \times \overrightarrow{P_0P_i} = \lambda\cdot\left(\mathbf{a}_g \times \mathbf{a}_g\right) = \vec{0}$$

- calculating contents with the means of the analytical geometry *(compare **figure 8**)*

$$\mathbf{A_{rectangle} = a \cdot b = a \cdot b \cdot \cos \sphericalangle(a,b)}$$

$$\mathbf{A_{parallelogram} = |a \times b| = a \cdot b \cdot \sin \sphericalangle(a,b)}$$

$$\mathbf{V_{parallelepiped} = |(a \times b) . c| = |(a \times b)| \cdot |b_a| \cdot \cos \sphericalangle(a \times b, c)}$$

- calculating contents with the means of elementary geometry (***figure 9***)

For rectangles of irrational sides lengths a calculation of the area by exhausting can only be done by calculation of an upper or lower limit by an improvement of the square grid as it is shown in the illustration on the next page.

Analogously the volume of a pyramid could be calculated. This idea could finally be transferred to develop the area formula for triangles (see to this /2/).
Analogous considerations with a geometrical background finally can also work as inspirations or starting points for complex tasks in mathematics. An example shall illustrate this.
The influence of the parameter n for linear functions of the shape y = mx+n is known to the pupils. It produces a sheaf of parallels (compare ***figure 10a***). This lead to the analogical conclusion that a "parallel" to the normal parable results from moving this along the y-axis. Graphical representation shows that this conclusion apparently was deceitful, if one does not want to differ from the "traditional" idea of parallelism (compare ***figure 10b***).
Further examination will show that parallelism has to be judged by curve increases. The analytical description of the wanted curve with the help of the tangent criterion for discreet points finally leads to the solution (compare ***figure 10c***).

Increase of the normal parable in the point P (1| 1)	m = 2
tangent equation	$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \lambda \in R$
orthogonality condition	$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \bar{x} - 1 \\ \bar{y} - 1 \end{pmatrix} = 0$
distance condition	$\sqrt{(\bar{x} - 1)^2 + (2 - 2\bar{x})^2} = 1$
point- coordinates (e.g.)	$\bar{P}(1 + \frac{1}{5}\sqrt{5} \mid 1 - \frac{2}{5}\sqrt{5})$
gradient in the point $\bar{P}$	$\bar{m} = 2$

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figure 1: using analogies while proving

$(a + b)^2 = a^2 + 2ab + b^2$

$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

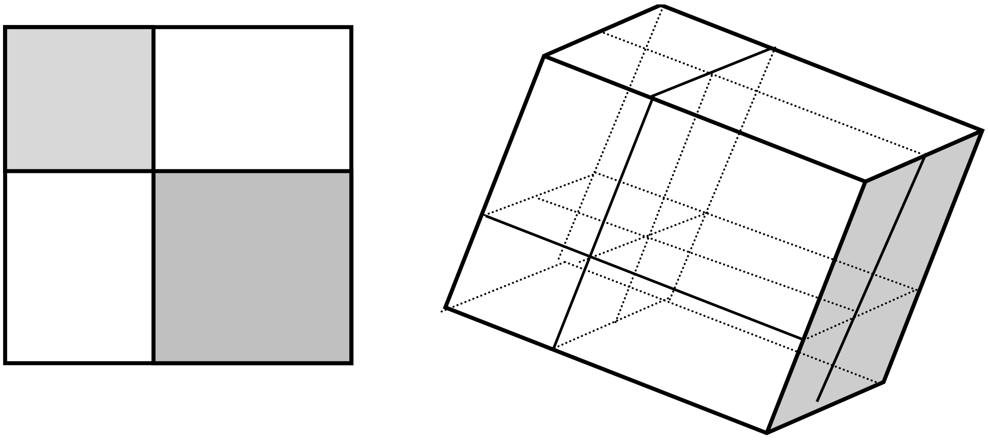


figure 2: a rectangle with a perimeter of 2a + 2b has to be changed into a square equal in perimeter

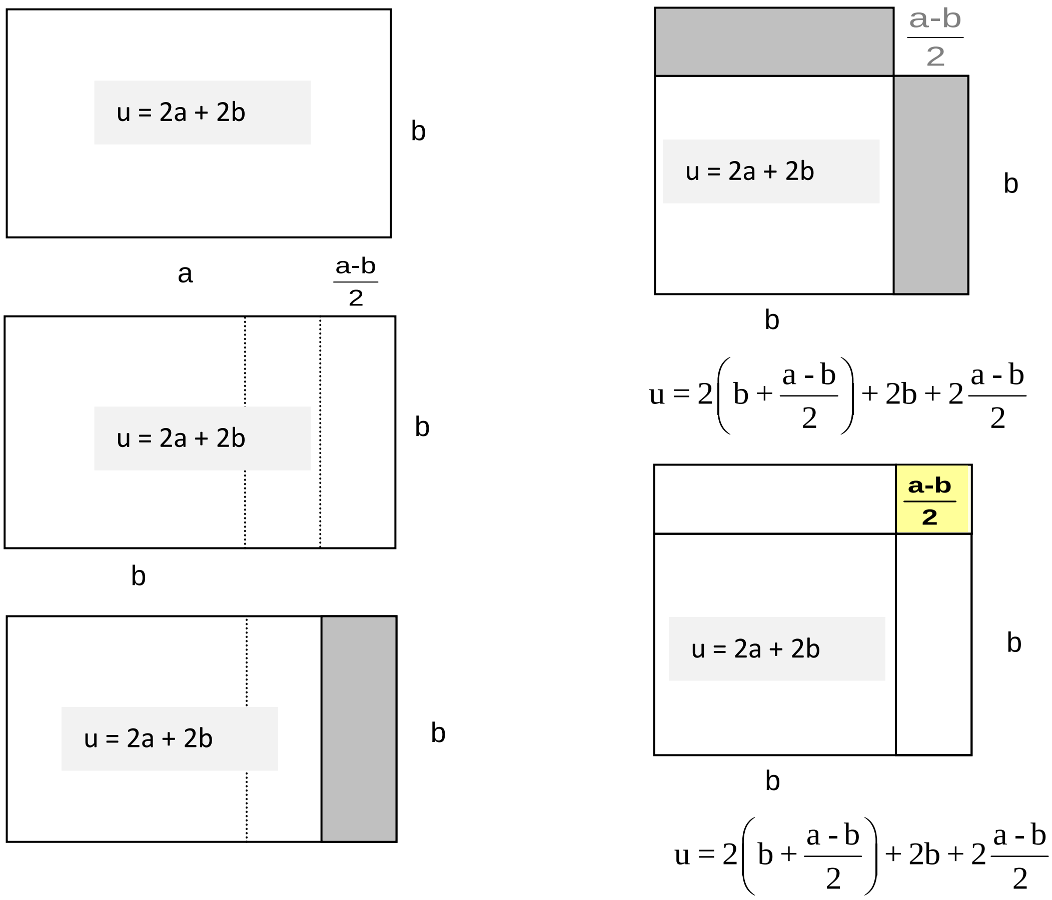


figure 3: a cuboid has to be changed into a cube equal in volume

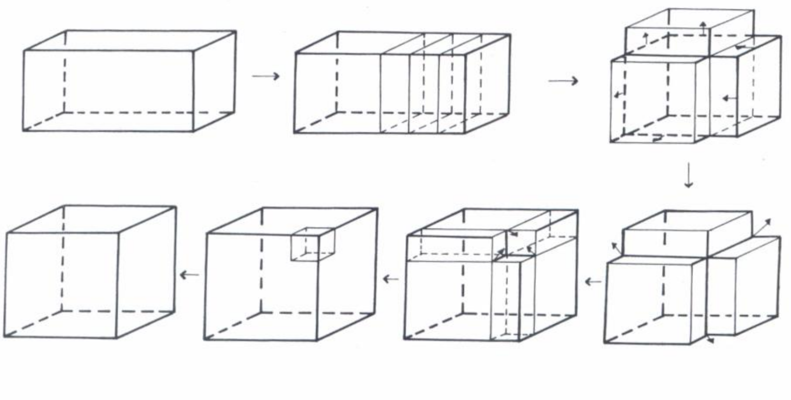


figure 4: the volume of a cylindrical vessel

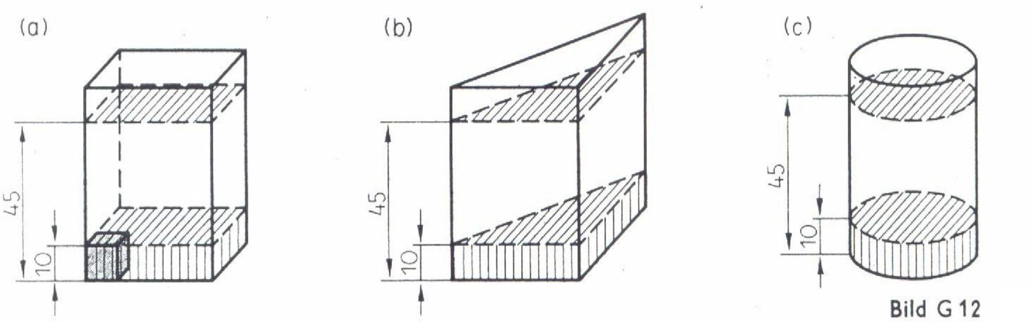


figure 5: the way tot he volume of a pyramid

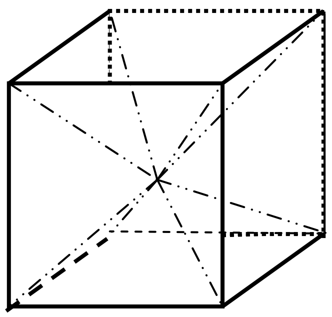


figure 6: the hypercube

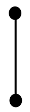
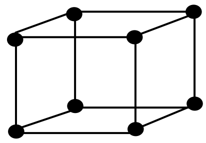
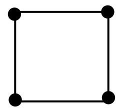
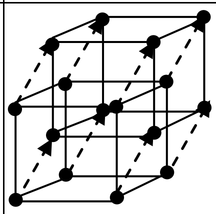
geometrical construction of the hypercube		|P| → |2P|		|Q| → |2Q + S|
		|S| → |2S + P|		|W| → |2W + Q|

figure 7: the theorems of proportional segments and their reversals

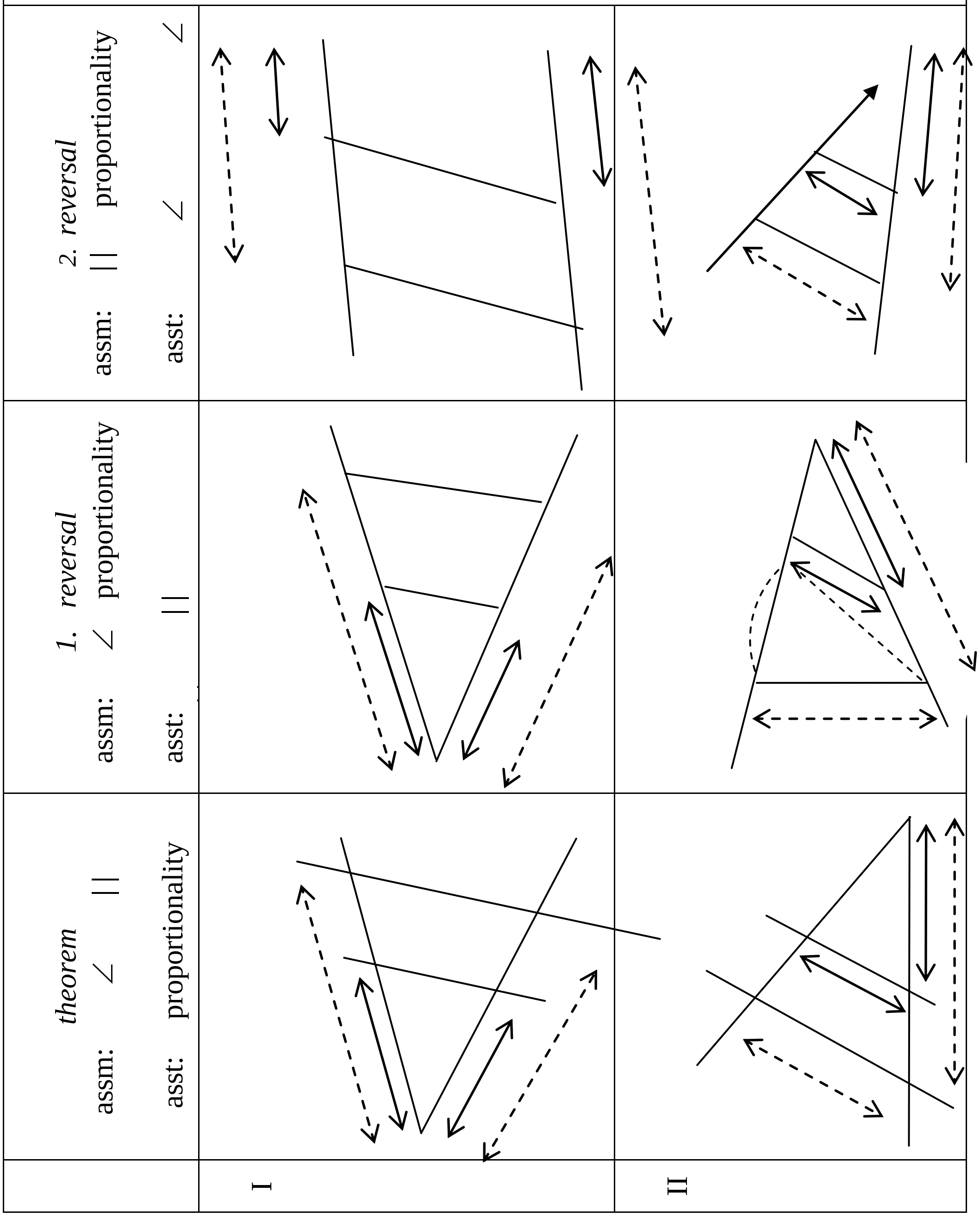


figure 8: calculating contents with the help of vector products

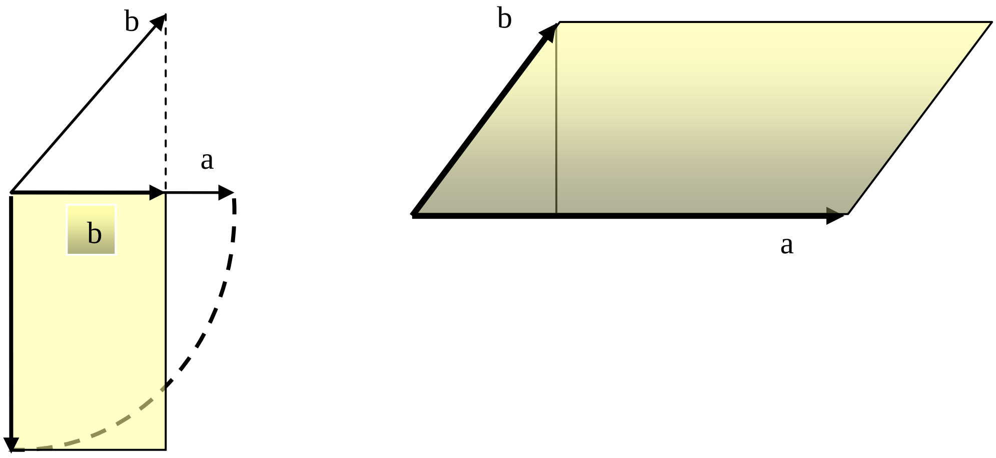
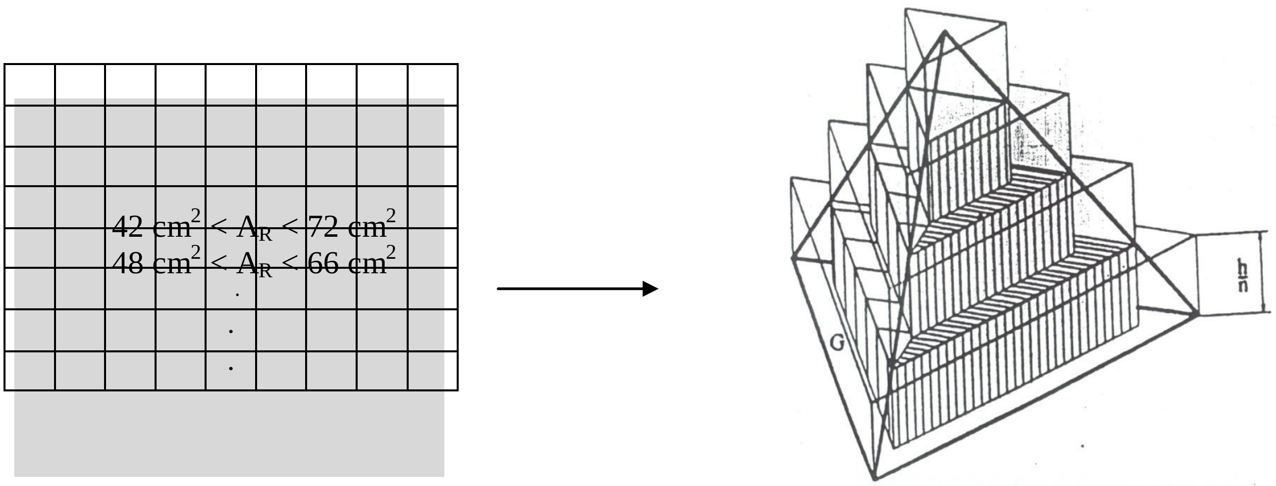


figure 9: the method of exhausition in different kinds



$$A_{\Delta(ABC)} = A_{P_1} + 2A_{D_1}$$
$$2A_{D_1} = 2A_{P_2} + 4A_{D_2}$$
$$4A_{D_2} = 4A_{P_3} + 8A_{D_3}$$

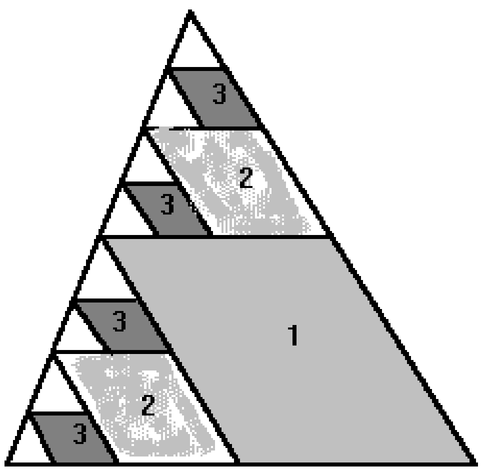


figure 10: parallelism as analogy

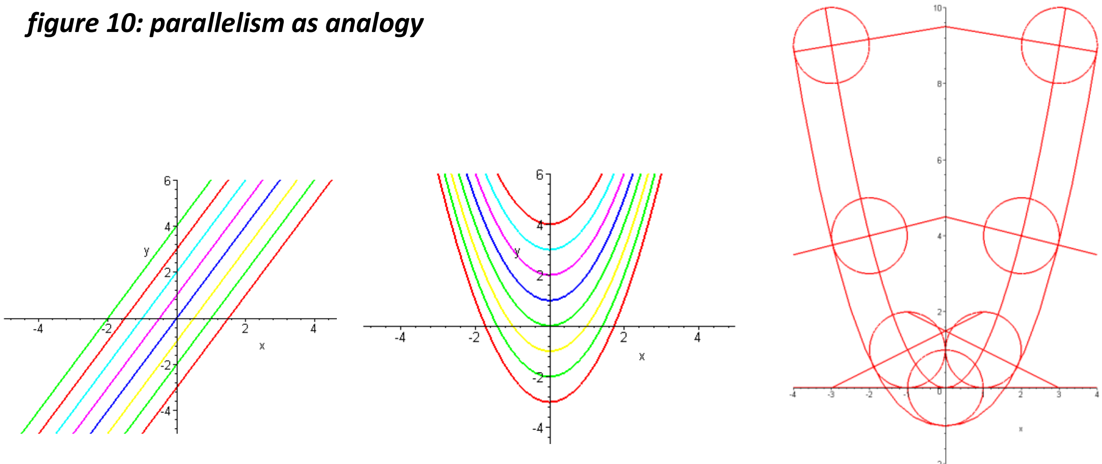


table 1: analogies by the pythagroen theorem

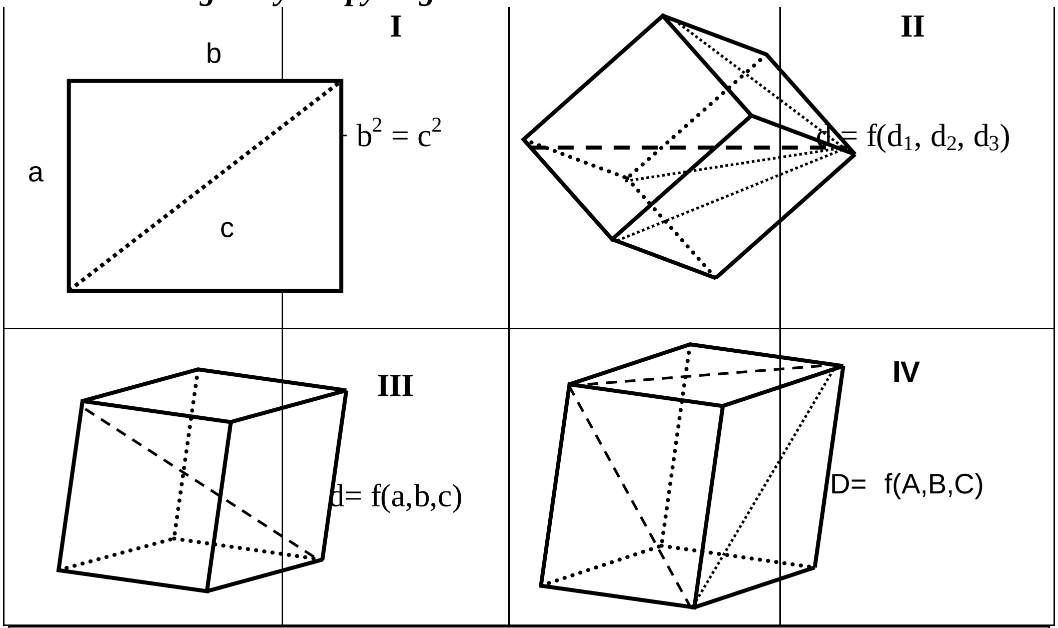


Table 2: analysis of the objects arissen by the analogies							
$\mathbb{R}^n$	points	segments	squares	cubes	hypercubes	h-h-cubes	Σ
0	1	0	0	0	0		1
1	2	1	0	0	0		3
2	4	4	1	0	0		9
3	8	12	6	1	0		27
4	16	32	24	8	1		81
5	32	80	80	40	10	1	243