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Muldowney Type Asymptotic Behaviors in the Study of Evolution Equations

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Abstract

Abstract. A particular case of dichotomy was introduced by J.S. Muldowney in [4], for linear differential systems. The aim of this paper is to define and characterize the concept of *Muldowney dichotomy* for skew-evolution semiflows. Connections of the new concept with the classic notion of exponential dichotomy in the uniform case are also given. We emphasize as well the importance of the dichotomy in the study of the solutions of evolution equations. Several illustrative examples motivate the approach. Connections with the classic notion of exponential dichotomy are also given. The approach is motivated by several illustrative examples. The study is performed in the uniform case.

Keywords: skew-evolution semiflow, exponential dichotomy, Muldowney dichotomy

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1. Introduction

As the state space of some dynamical systems that describe phenomena from physics, engineering or economics is of infinite dimension, the approach is more appropriate to be done by means of associated operator families. One of the most important asymptotic properties for the solutions of evolution equations is the exponential dichotomy, studied in the last years from various perspectives.

The techniques used for stability and instability can be generalized in the case of dichotomy, introduced by O. Perron in 1930 in [5], as a starting point for many papers on the stability theory, and which has gained importance since the works of J.L. Daleckii and M.G. Krein (see [1]), J.L. Massera and J.J. Schaffer (see [2]).

The study is led in this paper by means of skew-evolution semiflows on Banach spaces, defined by evolution semiflows and sevolution cocycles. They were introduced by us in [3] as generalizations of evolution operators and skew-product semiflows, the major difference consisting in the fact that a skew-evolution semiflow depends on three variables, while the classic concept of skew-product semiflow depends only on two. The skew-evolution semiflows are appropriate to study the asymptotic properties of the solutions for evolution equations having the form

$$\begin{cases} u'(t) = A(t)u(t), t > t_0 \geq 0 \\ \qquad u(0) = u_0. \end{cases} \tag{1.1}$$

where $A : \boldsymbol{R} \to B(V)$ denotes an operator with the properties $DomA(t) \subset V$ and $u_0 \in DomA(t_0)$.

Various concepts for the asymptotic properties such as stability, instability, dichotomy and trichotomy are studied in [6] and [7] for case of skew-evolution semiflows.

2. Definitions. Examples

We will consider a metric space *(X, d)*, a Banach space *V* and *B(V)* the space of all *V*-valued bounded operators defined on *V*. We will denote $Y = X \times V$ and $T = \{(t,s) \in \boldsymbol{R}_+^2 \mid t \geq s \geq 0\}$. The norm of vectors on *V* and of operators on *B(V)* is denoted by $\|\cdot\|$ and *I* is the identity operator on *V*.

Definition 2.1. A *skew-evolution semiflow* on *Y* is a mapping

$C : T \times Y \to Y$, defined by the relation $C(t,s,x,v) = (\varphi(t,s,x), \Phi(t,s,x)v)$,

where $\varphi : T \times X \to X$ with the properties:

(s_1) $\varphi(t,t,x) = x$, $\forall (t,x) \in \boldsymbol{R}_+ \times X$;

(s_2) $\varphi(t,s,\varphi(s,t_0,x)) = \varphi(t,t_0,x)$, $\forall (t,s),(s,t_0) \in T$, $\forall x \in X$

is an *evolution semiflow* on *X* and $\Phi : T \times X \to B(V)$ satisfying the conditions:

(c_1) $\Phi(t,t,x) = I$, $\forall (t,x) \in \boldsymbol{R}_+ \times X$;

(c_2) $\Phi(t,s,\varphi(s,t_0,x))\Phi(s,t_0,x)=\Phi(t,t_0,x)$, $\forall(t,s),(s,t_0)\in T$, $\forall x\in X$
is an *evolution cocycle* over φ.

Remark 2.2. The mapping $C_\lambda : T\times Y\to Y$, defined by

$$C_\lambda(t,s,x,v)=(\varphi(t,s,x),\Phi_\lambda(t,s,x)v),$$

where φ is an evolution semiflow, Φ is an evolution cocycle over φ and $\Phi_\lambda(t,s,x)=e^{-\lambda(t-s)}\Phi(t,s,x)$, is also a skew-evolution semiflow, called the λ *-shifted skew-evolution semiflow.*

In the following example we will give a skew-evolution semiflow that depends on all its variables and is generated by an evolution equation, as, for example, (1.1).

Example 2.3. Let $\boldsymbol{C}$ = $C(\boldsymbol{R}, \boldsymbol{R})$ be the metric space of all continuous functions $x:\boldsymbol{R}\to\boldsymbol{R}$, with the topology of uniform convergence on compact subsets of $\boldsymbol{R}$. We denote by X the closure in $\boldsymbol{C}$ of the set $\{f_t, f_t(s) = f(t+s), t,s\in\boldsymbol{R}_+\}$. Then (X, d), where

$$d(x,y)=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{\sup\limits_{t\in[0,n]}|x(t)-y(t)|}{1+\sup\limits_{t\in[0,n]}|x(t)-y(t)|},$$

is a metric space, and the mapping $\varphi : T\times X\to X$, $\varphi(t,s,x)=x_{t-s}$ is an evolution semiflow on X. Let $V=\boldsymbol{R}^2$ with the norm $\|v\|=|v_1|+|v_2|$, where $v=(v_1,v_2,v_3)\in V$. The mapping $\Phi : T\times X\to B(\boldsymbol{R})$

$$\Phi(t,s,x)v=\left(v_1 e^{\alpha_1\int_s^t x(\tau-s)d\tau}, v_2 e^{\alpha_2\int_s^t x(\tau-s)d\tau}\right),\ (\alpha_1,\alpha_2)\in\boldsymbol{R}^2,$$

is an evolution cocycle over the evolution semiflow φ and, hence, $C=(\varphi,\Phi)$ is a skew-evolution semiflow.

3. Muldowney type asymptotic behaviors

The main idea in the definition of the property of dichotomy for evolution equations is to obtain, at any moment and by means of properly chosen projectors, the decomposition of the state space into two closed subspaces: the stable subspace and the instable one. Therefore let us consider first the properties of uniform exponential stability and instability defined by us for skew-evolution semiflows.

Definition 3.1. A skew-evolution semiflow $C=(\varphi,\Phi)$ is said to be *uniformly exponentially stable* if there exist some constants $N\geq 1$, $\nu>0$ such that the relation:

$$e^{\nu(t-s)}\|\Phi(t,t_0,x)v\|\leq N\|\Phi(s,t_0,x)v\|$$

holds for all $(t,s),(s,t_0)\in T$ and for all $(x,v)\in Y$.

Definition 3.2. A skew-evolution semiflow $C=(\varphi,\Phi)$ is said to be *uniformly exponentially instable* if there exist some constants $N\geq 1$, $\nu>0$ such that the relation:

$$e^{\nu(t-s)}\|\Phi(s,t_0,x)v\|\leq N\|\Phi(t,t_0,x)v\|$$

holds for all $(t,s),(s,t_0)\in T$ and for all $(x,v)\in Y$, where Q denotes the complementary of the projections family P.

In order to introduce the definition of uniform exponential dichotomy, let us consider

Definition 3.3. A mapping $P: X\to B(V)$ with the property that $P(x)^2=P(x)$, for all $x\in X$, is called *projections family* on V. The mapping $Q: X\to B(V)$, defined by $Q(x) = I - P(x)$, is also a projections family, called the *complementary* of P.

Definition 3.4. A projections family $P: X\to B(V)$ is said to be compatible with a skew-evolution semiflow $C=(\varphi,\Phi)$ if the relation:

$$P(\varphi(t,s,x))\Phi(t,s,x)=\Phi(t,s,x)P(x)$$

holds for $(t,s)\in T$ and all $x\in X$.

We will denote $\Phi_P(t,s,x)=\Phi(t,s,x)P(x)$ and $C_P=(\varphi,\Phi_P)$. We have

(i) $\Phi_P(t,t,x)=P(x),\ \forall(t,x)\in \boldsymbol{R}_+\times X$

(ii) $\Phi_P(t,s,\varphi(s,t_0,x))\Phi_P(s,t_0,x)=\Phi_P(t,t_0,x)$, for all $(t,s),(s,t_0)\in T$ and for all $(x,v)\in Y$.

In what follows, let us consider a projections family $P: X\to B(V)$ and let us denote by Q the complementary of the projections family P.

Definition 3.5. A skew-evolution semiflow $C=(\varphi,\Phi)$ is said to be *uniformly exponentially dichotomic* relative to P if:

(ued_1) Φ_P is uniformly exponentially stable;

(ued_2) Φ_Q is uniformly exponentially instable.

An equivalent definition is given by

Proposition 3.6. *A skew-evolution semiflow $C=(\varphi,\Phi)$ is uniformly exponentially dichotomic if and only if there exist some constants $N\geq 1$, $\nu>0$ such that:*

(ued_1)’ $e^{\nu(t-s)}\|\Phi_P(t,s,x)v\|\leq N\|P(x)v\|$;

(ued_2)’ $e^{\nu(t-s)}\|Q(x)v\|\leq N\|\Phi_Q(t,s,x)v\|$,

for all $(t,s)\in T$ and for all $(x,v)\in Y$.

Proof. *Necessity.* It is immediate if we consider $s = t_0$ in Definition 3.1 Definition 3.2 and Definition 3.5.

Sufficiency. According to Definition 2.1 (c_2) and to the hypothesis, we have

$$\|\Phi_P(t,t_0,x)v\| = \|\Phi_P(t,s,\varphi(s,t_0,x))\Phi_P(s,t_0,x)v\| \le Ne^{-\nu(t-s)}\|\Phi_P(s,t_0,x)v\|$$

and

$$Ne^{-\nu(t-s)}\|\Phi_Q(t,t_0,x)v\| = Ne^{-\nu(t-s)}\|\Phi_Q(t,s,\varphi(s,t_0,x))\Phi_Q(s,t_0,x)v\| \ge$$
$$\ge \|\Phi_Q(s,t_0,x)v\|$$

for all $(t,s),(s,t_0)\in T$ and all $(x,v)\in Y$, which shows that C is a unifomlly exponentially dichotomic and ends the proof. □

Example 3.7. Let us consider the evolution semiflow φ on X given as in Example 2.3 and V = $\boldsymbol{R}^2$. Let $f:\boldsymbol{R}_+\to\boldsymbol{R}_+^*$ be a decreasing function with the property that there exists $\lim_{t\to\infty} f(t)=l>0$. We denote $\mu>f(0)$. The mapping $\Phi: T\times X\to B(V)$,

$$\Phi(t,s,x)(v_1,v_2,v_3) = \left(v_1 e^{-\mu(t-s)+\int_s^t x(\tau-s)d\tau}, v_2 e^{\int_s^t x(\tau-s)d\tau}\right)$$

is an evolution cocycle. We consider the projections families *P, Q: X→B(V)* *P(x)v = (v₁,0), Q(x)v = (0,v₂)*. Following inequalities hold

$$\|\Phi_P(t,t_0,x)v\| \le e^{[-\mu+x(0)](t-s)}\|\Phi_P(s,t_0,x)v\|;$$
$$\|\Phi_Q(t,t_0,x)v\| \ge e^{l(t-s)}\|\Phi_Q(s,t_0,x)v\|,$$

for all $(t,s),(s,t_0)\in T$, $(x,v)\in Y$, which proves that $C=(\varphi,\Phi)$ is uniformly exponentially dichotomic with $N=1$ and $\nu=\max\{\mu-x(0),l\}$.

Definition 3.8. A skew-evolution semiflow $C=(\varphi,\Phi)$ is said to be *Muldowney exponentially stable* if there exist some constants $N\ge 1$ and a mapping $\nu:\boldsymbol{R}_+\to\boldsymbol{R}_+$ such that following relation:

$$\|\Phi(t,t_0,x)v\| \le Ne^{-\int_s^t \nu(\tau)d\tau}\|\Phi(s,t_0,x)v\|;$$

holds for all $(t,s),(s,t_0)\in T$ and for all $(x,v)\in Y$.

Definition 3.9. A skew-evolution semiflow $C=(\varphi,\Phi)$ is said to be *Muldowney exponentially instable* if there exist a constant $N\ge 1$ and a mapping $\nu:\boldsymbol{R}_+\to\boldsymbol{R}_+$ such that following relation:

$$\|\Phi(s,t_0,x)v\| \le Ne^{-\int_s^t \nu(\tau)d\tau}\|\Phi(t,t_0,x)v\|,$$

holds for all $(t,s),(s,t_0)\in T$ and for all $(x,v)\in Y$.

Definition 3.10. A skew-evolution semiflow $C=(\varphi,\Phi)$ is said to be *Muldowney exponentially dichotomic* relative to P if:

(Med$_1$) Φ_P is Muldowney exponentially stable;

(Med$_2$) Φ_Q is Muldowney exponentially instable.

Remark 3.11. (i) If $Q = 0$, we obtain the asymptotic property of Muldowney exponential stability;
(ii) If $P = 0$, we obtain the asymptotic property of Muldowney exponential instability.

Proposition 3.12. *A skew-evolution semiflow* $C=(\varphi,\Phi)$ *is Muldowney exponentially dichotomic if and only if there exist a constant* $N \geq 1$ and a mapping $\nu : \boldsymbol{R}_+ \to \boldsymbol{R}_+$ *such that:*

(Med$_1$)' $\left\|\Phi_P(t,s,x)v\right\| \leq Ne^{-\int_s^t \nu(\tau)d\tau}\left\|P(x)v\right\|$;

(Med$_2$)' $\left\|Q(x)v\right\| \leq Ne^{-\int_s^t \nu(\tau)d\tau}\left\|\Phi_Q(t,s,x)v\right\|$,1

for all $(t,s)\in T$ *and for all* $(x,v)\in Y$.

Proof. Necessity. It is immediate if we consider $s = t_0$ in Definition 3.8 Definition 3.9 and Definition 3.10.
Sufficiency. According to Definition 2.1 (c$_2$) and to the hypothesis, we have

$$\left\|\Phi_P(t,t_0,x)v\right\| = \left\|\Phi_P(t,s,\varphi(s,t_0,x))\Phi_P(s,t_0,x)v\right\| \leq Ne^{-\int_s^t \nu(\tau)d\tau}\left\|\Phi_P(s,t_0,x)v\right\|$$

and

$$Ne^{-\int_s^t \nu(\tau)d\tau}\left\|\Phi_Q(t,t_0,x)v\right\| = Ne^{-\int_s^t \nu(\tau)d\tau}\left\|\Phi_Q(t,s,\varphi(s,t_0,x))\Phi_Q(s,t_0,x)v\right\| \geq$$
$$\geq \left\|\Phi_Q(s,t_0,x)v\right\|$$

for all $(t,s),(s,t_0)\in T$ and all $(x,v)\in Y$, which shows that C is unifomlly exponentially dichotomic and ends the proof.

A connection between the notions of uniform exponential dichotomy and Muldowney exponential dichotomy is given by □

Proposition 3.13. *If* $C=(\varphi,\Phi)$ *is uniformly exponentially dichotomic, then it is Muldowney exponentially dichotomic.*

Proof. As C is uniformly exponentially dichotomic, then, according to Definition 3.5, there exist some constants $N \geq 1$, $\nu > 0$ such that relations (ued$_1$) and (ued$_2$) hold for all $(t,s),(s,t_0)\in T$, $(x,v)\in Y$. We have

.

$$\left\|\Phi_P(t,t_0,x)v\right\| \leq Ne^{-\nu(t-s)}\left\|\Phi_P(s,t_0,x)v\right\| \leq Ne^{-\int_s^t \nu d\tau}\left\|\Phi_P(s,t_0,x)v\right\|$$

and

$$\left\|\Phi_Q(s,t_0,x)v\right\| \leq Ne^{-\nu(t-s)}\left\|\Phi_Q(t,t_0,x)v\right\| \leq Ne^{-\int_s^t \nu d\tau}\left\|\Phi_Q(t,t_0,x)v\right\|$$

for all $(t,s),(s,t_0)\in T$ and all $(x,v)\in Y$, which proves the statements (Med_1) and (Med_2) of Definition 3.10 and which shows that C is Muldowney exponentially dichotomic. □

The converse is not true, as shown in

Example 3.14. Let us consider the elements defined in Example 2.3 and Example 3.7. We consider the functions $u,w:\boldsymbol{R}_+\to\boldsymbol{R}$, given by

$$u(t)=e^{2t-t\sin t} \text{ and } w(t)=e^{3t-2t\cos t}$$

We define the mapping $\Phi : T\times X\to B(V)$ by

$$\Phi(t,s,x)v=\left(\frac{u(s)}{u(t)}v_1e^{-\int_s^t x(\tau-s)d\tau},\frac{w(t)}{w(s)}v_2e^{\int_s^t x(\tau-s)d\tau}\right),$$

which is an evolution cocycle over the evolution semiflow φ, whjch is Muldowney dichotomic but is not uniformly exponentially dichotomic.

Bibliography

Daleckii, J., Krein, M. (1974). ‘Stability of differential equations in Banach spaces’, *Amer. Math. Soc.*, Providence, RI, 43.

Massera, J.L., Sch\"{a}ffer, J.J. (1966). ‘Linear differential equations and function spaces’, *Pure Appl. Math.*, Academic Press, New York-London, 21.

Megan, M., Stoica, C. (2008). ‘Exponential instability of skew-evolution semiflows in

Muldowney, J.S. (1984). ‘Dichotomies and asymptotic behaviour for linear differential systems’, *Trans. Amer. Math. Soc*., 56(2):465-484.

Perron, O. (1930). ‘Die Stabilitatsfrage bei Differentialgleichungen’, *Math. Z.*, 32:703-728.

Stoica, C. (2011). 'Uniform asymptotic behaviors for skew-evolution semiflows on Banach spaces', Mirton Publishing House, Timisoara.

Stoica C., Megan M. (2010). ‘On uniform exponential stability for skew-evolution semiflows on Banach spaces’, *Nonlinear Analysis*, 72(3-4):1305-1313.