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Two Stones Kill One Bird

Guanshen Ren, PhD
Professor
Math Department
The College of St. Scholastica
USA

Athens Institute for Education and Research
8 Valaoritou Street, Kolonaki, 10671 Athens, Greece
Tel: + 30 210 3634210 Fax: + 30 210 3634209
Email: info@atiner.gr URL: www.atiner.gr
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Guanshen Ren, PhD
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Abstract

In this article we present several examples to show nice connections among branches of mathematics. Through the examples, students not only learn the new material but also revisit the old concepts and theorems. This is one way to enhance their learning experience.

Contact Information of Corresponding author:

In our daily life, no one disputes that "one stone kills two birds" is better than "two stones kill one bird". But in mathematics, "two stones kill one bird", in many cases, is as good as "one stone kills two birds". From the mathematics point of view, "two stones kill one bird" means that we try to solve one problem by different methods that apply concepts, theorems, and ideas from other branches of mathematics. For example, some algebra problems may be solved by applying geometry, and geometry problems may be solved by trigonometric or algebraic methods. In this way, students can see nice connections among mathematical branches, and learn the new while revisiting the old. To reach that goal, it is important to encourage current and future high school math teachers to explore this idea. In this article, we present some examples to show how to use this phenomenon in a math class for secondary math education students.

Example 1. Trigonometry and the DeMoivre's Theorem "kill" one geometry problem: Three squares are arranged together as in figure 1, prove that $\angle AFG + \angle AEF = \frac{\pi}{4}$.

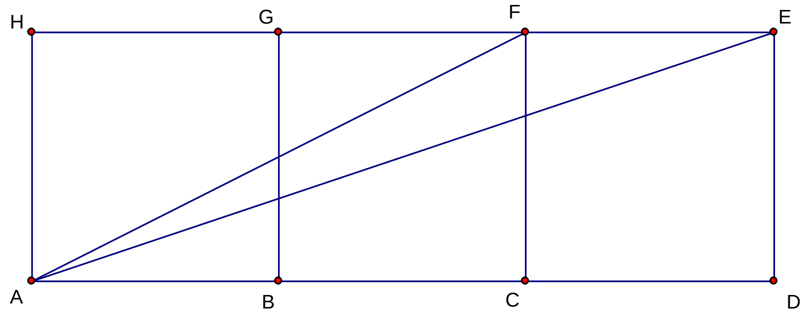

Figure 1

There are a few geometric solutions, but if we look at the problem from a trigonometric point of view, we end up with an interesting solution. As in **figure 1**, we assume that the side of each square is 1 unit, then

$AF = \sqrt{2^2 + 1^2} = \sqrt{5},\ AE = \sqrt{3^2 + 1^2} = \sqrt{10}$

and $\sin \angle AFH = \frac{1}{\sqrt{5}},\ \sin \angle AEH = \frac{1}{\sqrt{10}},\ \cos \angle AFH = \frac{2}{\sqrt{5}},\ \cos \angle AEH = \frac{3}{\sqrt{10}}$.

Now

$$\cos(\angle AFH + \angle AEH) = \cos \angle AFH \cdot \cos \angle AEH - \sin \angle AFH \cdot \sin \angle AE$$

$$= \frac{2}{\sqrt{5}} \cdot \frac{3}{\sqrt{10}} - \frac{1}{\sqrt{5}} \cdot \frac{1}{\sqrt{10}} = \frac{5}{5\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

Since both $\angle AFH$ and $\angle \mathrm{AEH}$ are acute angles, $\angle AFH + \angle AEH = \frac{\pi}{4}$.

Second method is to apply the DeMoivre's theorem for complex numbers. We establish a *xy*- coordinate system as follows: let point *A* be the origin and *AD*, *AH* be the *x*-axes and *y*-axes, the length of *AB* be the unit. Then points *F* and *G* represent complex numbers $2+i, 3+i$, and $\angle AFG$ and $\angle AEF$ are the arguments of $2+i$, $3+i$ respectively.

$\angle AFG + \angle AEF$ is the argument of the product of $(2+i)$ and $(3+i)$ which is $5+5i$.

The argument of $5+5i$ is $\frac{\pi}{4}$ since (5, 5) is a point in quadrant I. Thus

$\angle AFG + \angle AEF = \frac{\pi}{4}$.

Example 2. One traditional way to prove trigonometric identities is applying basic trig identities. On the other hand, some trigonometry identities may be informally "proved" by geometric figures as illustrated in the book "*Proofs Without Words*". Here we present one more such geometric figure to "prove" two trig identities without applying other trig identities.

A. A half-angle identity for tangent: $\tan x = \frac{\sin 2x}{1+\cos 2x}$; **Figure 2.1** illustrates the identity by letting $\angle D = 90^\circ$, $AB = BC = 1$. The critical steps in construction of such triangle is to be carefully assign number one and the angle *x* to parts of a right triangle to get some trig functions in the trig identity. If not all of them show up in the construction, try to add something to obtain the other desired trig functions. Then we may apply some geometric concepts such as similarity of triangles to finish the job. In this example, we construct a right triangle *ABD* first with the assumption that angle *x* is less than $\frac{\pi}{4}$ and $\angle ABD = 2x$, $AB = 1$. Thus we have $AD = \sin 2x$, $BD = \cos 2x$. But we still need a $\tan x$, that's why we add an isosceles triangle *ABC* to the picture. Now $\tan x = \frac{AD}{CD} = \frac{\sin 2x}{1+\cos 2x}$.

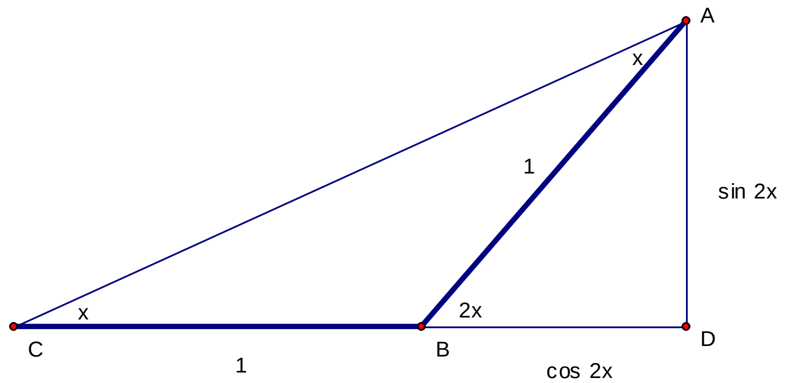

Figure 2.1

B. The double-angle identities: $\sin 2x = 2\sin x \cos x$; $\cos 2x = 2\cos^2 x - 1$.

We just simply draw *BE* perpendicular to *AC* in **Figure 2.1** to get both sin *x* and cos *x*. Using the similarity of triangles *ADC* and *BEC* as in **figure 2.2**, we have $\frac{\sin 2x}{\cos x + \cos x} = \frac{\sin x}{1}$, *i.e.* $\sin 2x = 2\sin x \cos x$.

$\frac{1+\cos 2x}{\cos x + \cos x} = \frac{\cos x}{1}$, *i.e.* $\cos 2x = 2\cos^2 x - 1$.

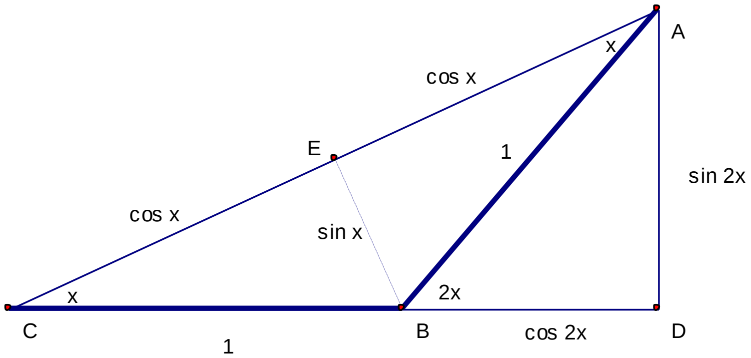

Figure 2.2

Geometry figures may not only demonstrate well-known trig identities but also "prove" some other trig identities. We pick one exercise from the textbook "*College Algebra with Trigonometry*" by Barnett, Ziegler, Byleen, in 2007.

Example 3. $\frac{\csc x}{\cos x} = \cot x + \tan x$;

Construct right triangle BCD such that $\angle BCD = x, CD = 1$. Now we have $BD = \tan x$. To get the rest of trig functions involved, we draw $DE \perp BC, AC // DE$. Then $CE = \cos x, AC = \csc x, AD = \cot x$ and $AB = \tan x + \cot x$. Triangles ABC and CDE are similar, $\frac{AC}{CE} = \frac{AB}{CD}$ gives the desired result.

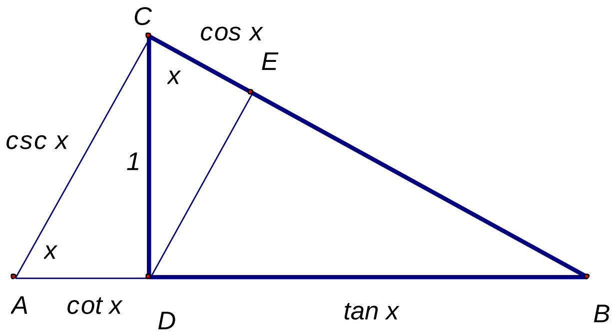

Figure 3

Example 4.

If a, b, c, X, Y, Z are positive real numbers such that $a + X = b + Y = c + Z = k$, then $aY + bZ + cX < k^2$.

An algebraic proof needs to multiply $a + X = k, b + Y = k, c + Z = k$ together, and then do some manipulations to get the inequality. Here we present a graphic illustration that makes the inequality easy and clear as in **figure 4** .

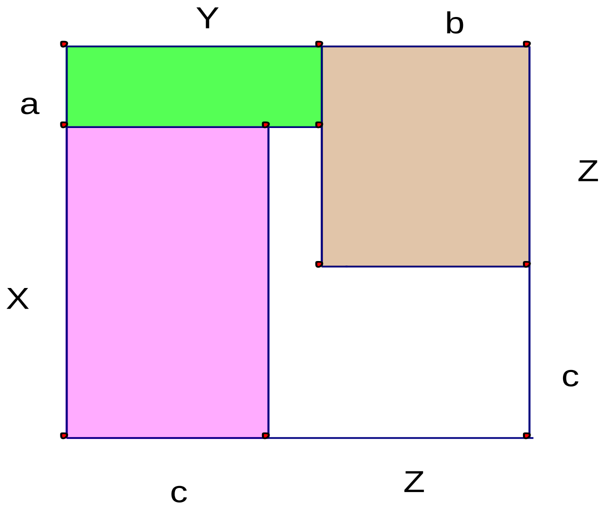

Figure 4

Example 5. In triangle *ABC, AB* = *AC, BE* bisects angle *B*, $\angle A = 100^\circ$. Prove that $BC = AE + BE$.

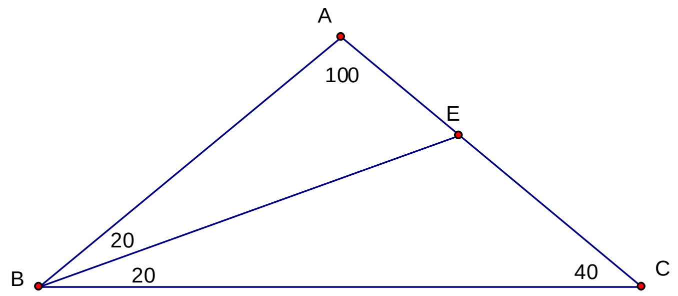

Figure 5

One geometric proof would require drawing *BF* on *BC* such that *BF* = *BE*, then using similar triangles and a property of angle bisector to finish the proof. Here we present a proof by applying trigonometry.

Proof: For convenience, we may assume that *AB* = *AC* = 1 in **figure 5**. Then $\frac{BE}{\sin 100^\circ} = \frac{1}{\sin 60^\circ} = \frac{AE}{\sin 20^\circ}$, and $BE = \frac{\sin 100^\circ}{\sin 60^\circ}$, $AE = \frac{\sin 20^\circ}{\sin 60^\circ}$. Also $\frac{BC}{\sin 100^\circ} = \frac{1}{\sin 40^\circ}$, *i.e.* $BC = \frac{\sin 100^\circ}{\sin 40^\circ}$, so to prove *BC* = *AE* + *BE* is to prove

$$\frac{\sin 100^\circ}{\sin 40^\circ} = \frac{\sin 100^\circ + \sin 20^\circ}{\sin 60^\circ} \qquad (1).$$

But $\sin 100^\circ = \sin 80^\circ = 2\sin 40^\circ \cos 40^\circ$, $\frac{\sin 100^\circ}{\sin 40^\circ} = \frac{\sin 80^\circ}{\sin 40^\circ} = 2\cos 40^\circ$.

Therefore identity (1) is equivalent to

$$2\sin 60^\circ \cos 40^\circ = \sin 100^\circ + \sin 20^\circ \qquad (2)$$

Note that $\sin 100^\circ = \sin(60^\circ + 40^\circ) = \sin 60^\circ \cos 40^\circ + \cos 60^\circ \sin 40^\circ$, thus to prove identity (2) is equivalent to show $\sin 60^\circ \cos 40^\circ = \sin 20^\circ + \cos 60^\circ \sin 40^\circ$, which is true from the trig identity $\sin 20^\circ = \sin(60^\circ - 40^\circ) = \sin 60^\circ \cos 40^\circ - \cos 60^\circ \sin 40^\circ$.

The two phenomena of "one stone kills two birds" and "two stones kill one bird" do not contradict to each other in mathematics. In fact both of them are very helpful in teaching and learning math. We introduce above examples to show the beautiful connections among branches of math to enhance deeper understanding of math for students, and also use them as an "appetizer" so that others may come up with new, interesting examples that will inspire students to do more exploration in mathematics.

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