Athens Institute for Education and Research
ATINER

ATINER's Conference Paper Series
MAT2012-0103

On Millionschikov's Zero-Fourth Cumulant Hypothesis Applied to Turbulence

H. P. Mazumdar
Indian Statistical Institute
Calcutta – 700035
India

C. Mamaloukas
Department of Statistics
Athens University of Economics and Business
Greece

Athens Institute for Education and Research
8 Valaoritou Street, Kolonaki, 10671 Athens, Greece
Tel: + 30 210 3634210 Fax: + 30 210 3634209
Email: info@atiner.gr URL: www.atiner.gr
URL Conference Papers Series: www.atiner.gr/papers.htm

Printed in Athens, Greece by the Athens Institute for Education and Research.

ISSN **2241-2891**
5/09/2012

An Introduction to ATINER's Conference Paper Series

ATINER started to publish this conference papers series in 2012. It includes only the papers submitted for publication after they were presented at one of the conferences organized by our Institute every year. The papers published in the series have not been refereed and are published as they were submitted by the author. The series serves two purposes. First, we want to disseminate the information as fast as possible. Second, by doing so, the authors can receive comments useful to revise their papers before they are considered for publication in one of ATINER's books, following our standard procedures of a blind review.

Dr. Gregory T. Papanikos
President
Athens Institute for Education and Research

This paper should be cited as follows:

Mazumdar, H. P. and Mamaloukas, C. (2012) "**On Millionschikov's Zero-Fourth Cumulant Hypothesis Applied to Turbulence**" Athens: ATINER'S Conference Paper Series, No: MAT2012-0103.

On Millionschikov's Zero-Fourth Cumulant Hypothesis Applied to Turbulence

H. P. Mazumdar
Indian Statistical Institute
Calcutta – 700035
India

C. Mamaloukas
Department of Statistics
Athens University of Economics and Business
Greece

Abstract

Abstract: Quasi-normality hypothesis or zero-fourth cumulant approximation is generally applied for the closure of turbulence problems. This hypothesis is due to well known scientist M.D. Millionschikov. It has been applied to solve various turbulence problems of theoretical and applied interest e.g. Incompressible turbulence velocity and temperature fields, hydrodynamic and hydromagnetic pressure fluctuations, magneto-hydrodynamic turbulence etc. In this paper, we intend to seek further possible improvements of this idea from application points of view.

Contact Information of Corresponding author:

1. Introduction

The inherent difficulty with the closure problem of turbulence is well known. In tackling this problem, generally some physically based hypotheses are often constructed. Such a hypothesis was first forwarded by Chow (1940) and Millionschikov (1941) in which an assumption was made that the fourth order velocity correlation is related to second order correlation in the same way as for a normal joint probability distribution of the velocity. Final version of this statistical hypothesis is named after M.D. Millionschikov.

Uberoi (1953) conducted extensive measurements for some cases and showed that the discrepancy between the measured fourth order moments and the prediction deduced from Millionschikov's quasi-normality hypothesis are found to lie within the limits of experimental errors.

Ogura (1963) carried out numerical computation of the turbulence energy spectrum $E(\kappa,t)$ and showed that it becomes negative over a finite wave number range when Reynolds number becomes sufficiently large.

Mirable (1969) pointed out that the negative energy observed by Ogura is due to the assumption that at the initial time t=0 the third order moment of the velocity field is identically zero.

Millionschikov (1941) proposed that probability distributions of the velocities at two different points but at one and the same instant of time, are approximately in normal distribution and Uberoi (1953) verified this hypothesis experimentally.

Panchev (1971) discussed that Millionschikov's hypothesis may be extended further to the cases wherein quadruple moments are formed for one and the same instant of time but for more than two points.

Ogura (1963) discovered that the kinetic energy spectrum $E(\kappa,t)$ becomes negative over a finite wave number range if the Reynolds number is sufficiently large.

Proudman and Reid (1954) investigated that the joint probability distribution of fluctuations velocity components at three points is approximately normal. They showed that Loitsiansky's integral is not invariant of the motion and the first time derivative of triple correlation function, say $K(r)$ is proportioned to for large values of r.

Mirabel (1969) pointed out that the negative energy, obtained by Ogura (1969) is the outcome of the assumption that at the initiation time $t=0$, the third moment of the velocity field is identically zero. This implies that at the initial time there is no transfer of energy over the velocity spectrum.

Ghosh (1972) investigated the early-period decay process of a general type of turbulence using quasi-normality hypothesis and established two lemmas.

Lemma I: This lemma is concerned with the behavior of correlation tensors in the energy space when two or more points under reference coincide.

Let $\Psi_{i,j}\left(\vec{\kappa},\vec{\kappa'},t\right)$ and $\Psi_{i,j,\kappa}\left(\vec{\kappa},\vec{\kappa'},\vec{\kappa''},t\right)$ are spectrum functions which correspond respectively to correlation functions $F_{i,j}\left(x,x',t\right)\left[=\overline{u_i u'_j}\right]$ and $F_{i,j,\kappa}\left(x,x',x'',t\right)\left[=\overline{u_i u'_j u''_\kappa}\right]$. When the third point x'' merges with the first point x, then we have:

$$\int \Psi_{i,j,\kappa}\left(\overline{\lambda-\kappa''},\vec{\kappa'},\vec{\kappa''},t\right)d\vec{\kappa''}=\Psi_{ik,j}\left(\lambda,\vec{\kappa'},t\right) \tag{1.1}$$

where $\vec{\lambda}=\vec{\kappa}+\vec{\kappa'}$

Lemma II: It is concerned with the Millionschikov's quasi-normality hypothesis. We consider additional fluctuation velocity component u_l''' at the point x''', and the spectrum tensor $\Psi_{i,j,\kappa,l}\left(\vec{\kappa},\vec{\kappa'},\vec{\kappa''},\vec{\kappa'''},t\right)$ which correspond to correlation function $F_{i,j,\kappa,l}\left(x,x',x'',x''',t\right)\left[=\overline{u_i u'_j u''_\kappa u'''_l}\right]$.

When the fourth point x''' coincides with the first point x, we derive the relation

$$\begin{aligned}\Psi_{il,j,\kappa}\left(\vec{\kappa},\vec{\kappa'},\vec{\kappa''},t\right) &= \int \Psi_{i,j}\left(\overrightarrow{\kappa-\kappa'''},\vec{\kappa'},t\right)\Psi_{\kappa,l}\left(\vec{\kappa''},\vec{\kappa'''},t\right)d\vec{\kappa'''}\\ &+\int \Psi_{i,j}\left(\overrightarrow{\kappa-\kappa'''},\vec{\kappa''},t\right)\Psi_{j,l}\left(\vec{\kappa'},\vec{\kappa'''},t\right)d\vec{\kappa'''}\\ &+\int \Psi_{il}\left(\vec{\kappa},t\right)\Psi_{j,\kappa}\left(\vec{\kappa'},\vec{\kappa''},t\right)\end{aligned} \tag{1.2}$$

Now, if the third point x'' merges with the second point x', the following relation may be derived

$$\begin{aligned}\Psi_{il,j\kappa}\left(\vec{\kappa},\vec{\kappa'},t\right) = \iint\Big[&\Psi_{i,j}\left(\overrightarrow{\kappa-\kappa'''},\overrightarrow{\kappa'-\kappa''},t\right)\Psi_{\kappa,l}\left(\vec{\kappa''},\vec{\kappa'''},t\right)\\ &+\Psi_{i,\kappa}\left(\overrightarrow{\kappa-\kappa'''},\vec{\kappa''},t\right)\Psi_{j,l}\left(\overrightarrow{\kappa'-\kappa''},\vec{\kappa'''},t\right)\Big]d\vec{\kappa''}d\vec{\kappa'''}\\ &+\Psi_{il}\left(\vec{\kappa},t\right)\Psi_{j,\kappa}\left(\vec{\kappa'},t\right)\end{aligned} \tag{1.3}$$

In the next section, we would apply these lemmas to derive briefly, the Proudman-Reid type decay equations for the velocity, temperature and hydromagnetic fields.

Moreover, the relevant expressions for the fluctuating pressure fields under hydrodynamic and hydromagnetic conditions will also be stated for the sake of completeness.

2. Applications of quasi-normality hypothesis

a) Early-period decay process of turbulence

Ghosh (1972) showed that equation $\frac{\partial^2}{\partial t^2}\Psi_{i,j}\left(\vec{\kappa},\vec{\kappa'},t\right)=I_1\left(\vec{\kappa},\vec{\kappa'},t\right)+I_2\left(\vec{\kappa},\vec{\kappa'},t\right)$ is of the form for the general type of turbulence stated above and represents the early-period decay process. Here $I_1\left(\vec{\kappa},\vec{\kappa'},t\right)$ and $I_2\left(\vec{\kappa},\vec{\kappa'},t\right)$ have the requisite expressions. He derived the Proudman-Reid equation in this case.

b) Decay process of turbulence at large Reynolds and Peclet numbers:

Mazumdar (1976) studied the fluctuating temperature field considered to be superimposed on a general field of eddy turbulence by employing the quasi-normality hypothesis. In this investigation Mazumdar approached phenomenologically that the region under consideration is such that the variations of the mean temperature and mean velocity may be neglected because the transportation of thermal energy from place to place is very rapid. He derived the Proudman-Reid type equation of this case The equation is read as,

$$\frac{\partial^2}{\partial t^2}\int_0^\infty \kappa^2 E_{\theta\theta}\left(\kappa,t\right)d\kappa=\frac{4}{3}\int_0^\infty \kappa^2 E_{\theta\theta}\left(\kappa,t\right)d\kappa\cdot\int_0^\infty \kappa^2 F\left(\kappa,t\right)d\kappa$$

c) Mazumdar (2010) also derived the Proudman-Reid type decay equations for the magneto hydro-dynamic (MHD) turbulence when both eddy Reynolds number and the magnetic Reynolds number are very large.

d) Mazumdar (1979, 1984) applied the Millionschikov's quasi-normality hypothesis, respectively to the fields of hydro-dynamic and hydro magnetic turbulence and derived the expressions for the spectra of respective pressure fluctuations.

3. Refinement of Millionschikov's quasi-normality hypothesis:

The applications of Millionschikov's quasi-normality hypothesis are found to fail in some cases. For example, in case of convective boundary layer turbulence (Gryanik et.al. 2005) found that the probability density functions of temperature and vertical velocity fluctuations are skewed.

Losch (2004) devised parametrizations of fourth-order moments according to a universal model presented by (Gryanik et.al. 2005) and found to be more accurate than their corresponding Gaussian parametrizations which are based on Millionschikov's hypothesis.

Mirabel (1969) solved numerically, the system of moment equations e.g.

$$\left(\frac{\partial}{\partial t}+2\nu\kappa^2\right)E(\kappa,t)=\int_0^\infty\int_{\kappa-\kappa'}^{\kappa+\kappa'}F(\kappa,\kappa',\kappa'',t)\,d\kappa''\,d\kappa'$$

and

$$\left[\frac{\partial}{\partial t}+\nu\left(\kappa^2+\kappa'^2+\kappa''^2\right)\right]F(\kappa,\kappa',\kappa'',t)=\phi_1(\kappa,\kappa',\kappa'')E(\kappa,t)E(\kappa',t)+ \\ \phi_2(\kappa,\kappa',\kappa'')E(\kappa'',t)E(\kappa,t)+\phi_3(\kappa,\kappa',\kappa'')E(\kappa',t)E(\kappa'',t) \qquad (3.1)$$

where v is the kinematic viscosity, based on a finite difference method with proper initial conditions.

Conclusions

i) Millionschikov's quasi-normality hypothesis is considered to be very useful for closure of homogeneous and isotropic turbulence as such a hypothesis has been proved to be valid within the limits of experimental errors.
ii) To gain more insight into the closure problems of turbulence, data from recent and advanced level measurements are to be brought into account in developing the appropriate models.
iii) As and when necessary, modification of this hypothesis is welcomed, as Ogura (1963) has pointed that tentatively, the errors that arise from finite difference approximations in numerical integration of the main equation of turbulence are not responsible for the generation of the negative energy spectrum but are the consequences of the quasi-normality hypothesis itself.
iv) Third-order moments of the velocity field should not be assumed zero and it should be taken into account into the calculation even at the initial evolution of spectral energy of turbulence.
v) It is to be noted that the measured values of fourth-order mixed velocity-temperature moments in the atmospheric surface layer agree well with the quasi-gaussian assumption. Further, Schwarz inequalities, commonly used in the clipping approximation in turbulence modeling, are found to provide counts for third-order moments of ω, θ that are too conservative. These types of approaches are to be encouraged.

References

Batchelor G.K. and Proudman I. (1956), Phil.Trans.Roy.Soc. 248A, 167.

Chow W. L. (1940), *Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung*, J. Math. Ann., pp. 98-105.

Ghosh K.M. (1972), Indian Journal of Pure and Appl. Math.,Vol. 3, No.1, p. 157

Gryanik V.M. et.al. (2005), *A Refinement of the Millionshchikov Quasi-Normality Hypothesis for Convective Boundary Layer Turbulence*, J. Atmos. Sci., 62, pp. 2632–2638.

Losch Martin (2004), On the validity of the Millionshchikov quasi-normality hypothesis for open-ocean deep convection, Geophysical Research Letters, Vol. 31, L23301, 4 pp., doi:10.1029/2004GL021412.

Mazumdar H.P. (1976), *On the decay process of turbulence at large Reynolds and Peclet numbers*, App. Sci. Res., Vol. 32, p. 571.

Mazumdar H. P. (1979), *On the Pressure Fluctuations Associated with a General Type of Turbulence*, Appl. Sci. Res. Vol 35, No 5-6, pp 367-371.

Mazumdar H.P. (1984), *On the fluctuations of total pressure associated with general type of hydromagnetic turbulence.* Archeves of Mechanics, Vol. 36, pp. 233-240.

Mazumdar H.P. (2010), On Incompressible Hydromagnetic Turbulence, Research monograph), Calcutta Mathematical Society.

Millionschikov M. (1941), *On the theory of homogeneous isotropic turbulence*, Dokl . Akad. Nauk SSSR, 32, pp 615–618.

Mirabel A. P. (1969), *Application of Millionschikov's Hypothesis to the Problem of Isotropic Turbulence Degeneration*, Izv. AN SSSR. Mekhanika Zhidkosti i Gaza, Vol. 4, No. 5, pp. 171-175.

Ogura Y. (1963), *A Consequence of the Zero-Fourth-Cumulant Approximation in the Decay of Isotropic Turbulence*, Journal of Fluid Mechanics, vol. 16, pp 33-40.

Panchev S. (1971), *Random Functions and Turbulence* (Pergamom Press. Oxford), p. 444.

Proudman I. and Reid W.H. (1954), Phil.Trans.Roy.Soc. 247A, 163.

Uberoi M. S. (1953), Quadruple velocity correlations and pressure fluctuations in isotropic turbulence, J. Aeronaut. Sci., Vol. 20, p 197.