Athens Institute for Education and Research
ATINER

ATINER's Conference Paper Series
EMS2015-1779

Surprising Investigation of Loci Using Dynamic Software

Ruti Segal
Lecturer
Shaanan College, Oranim College
Israel

Moshe Stupel
Assistant Professor
Shaanan College, Gordon College
Israel

Victor Oxman
Senior Lecturer
Western Galilee College, Shaanan College
Israel

An Introduction to ATINER's Conference Paper Series

ATINER started to publish this conference papers series in 2012. It includes only the papers submitted for publication after they were presented at one of the conferences organized by our Institute every year. This paper has been peer reviewed by at least two academic members of ATINER.

Dr. Gregory T. Papanikos
President
Athens Institute for Education and Research

This paper should be cited as follows:

Segal, R., Stupel, M. and Oxman, V. (2016). **"Surprising Investigation of Loci Using Dynamic Software",** Athens: ATINER'S Conference Paper Series, No: **EMS2015-1779.**

Athens Institute for Education and Research
8 Valaoritou Street, Kolonaki, 10671 Athens, Greece
Tel: + 30 210 3634210 Fax: + 30 210 3634209 Email: info@atiner.gr URL: www.atiner.gr
URL Conference Papers Series: www.atiner.gr/papers.htm
Printed in Athens, Greece by the Athens Institute for Education and Research.
ISSN: 2241-2891
08/01/2016

Surprising Investigation of Loci Using Dynamic Software

Ruti Segal

Moshe Stupel

Victor Oxman

Abstract

The locus is a very important concept in Euclidean geometry since it serves as a tool for solving different problems, and allows geometric constructions to be carried out. The teaching of the subject of loci in various mathematics courses includes the solution of different exercises in which the student is required to find the locus in accordance with the data of the question. The present paper offers a different view of the subject of loci, which brings about the conceptual understanding of the subject with the identification of conserved properties and suitable generalizations obtained through investigation that includes the use of dynamic geometric software (Geogebra).

Keywords: combining technology and mathematics, dynamic geometric software, locus

Introduction

The locus is defined as a geometric shape, whose points, and only these points, share a common property. Or, using the language of set theory: a locus is the collection of all the points that satisfy a certain condition [1, 2].

In the traditional approach for studying loci, students are asked to prove that a certain curve is the locus that satisfies the requirements that are given in the problem. In mathematics, as an investigating science, the sought locus is usually not known beforehand, and one tries to guess its form based on experience, intuition, particular cases, computer applications, etc. In the end, one has to prove that the indicated locus really satisfies the requirements of the problem.

The use of dynamic geometric software allows the subject of the locus to be studied using a similar method to that of the mathematical researcher, allowing the student to carry out research-like work.

The use of dynamic software allows the students to solve problems by learning from examples. The students infer the essential steps from the examples, follow the critical properties of the concept, internalize them and subsequently implement them in the solution of the problems. The ability of the computer to generate many diverse examples fast, to save and repeat steps and to provide qualitative feedback, rather than just judgmental feedback, gives the student information on the mathematical concept that serves as a basis for generalizations and hypotheses that require proof [3-6].

There are different ways by which loci are formed. Using the following tasks will present different methods in which loci can be formed, where each example is dependent upon:

a) Showing the locus using an applet of dynamic software.
b) Giving mathematical proofs of the shape of a locus.

Bellow we will illustrate this approach to teaching about locus with several examples.

Task 1 – Loci Formed by the Points of Intersection of Special Lines in a Triangle Inscribed in a Circle

The triangle ΔABC is inscribed in a circle, such that the vertices of its base, B and C, are fixed, and its vertex A moves on the circular arc BC (see Figure 1). When point A moves to point A', the locus moves from point G to point G'. Find the locus for each of the following cases:

(a) What is the locus of the point of intersection of the **mid perpendiculars** in each of the triangles?
(b) What is the locus of the point of intersection of the **altitudes** in each of the triangles?
(c) What is the locus of the point of intersection of the **angle bisectors** in each of the triangles?
(d) What is the locus of the point of intersection of the **medians** in each of the triangles?

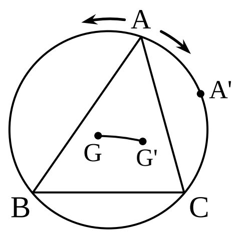

Figure 1

The Loci

Case (a)

It is well-known that the point of intersection of the mid perpendiculars in any triangle is the center of the circle that circumscribes it. In the present case all the triangles are circumscribed in the same circle, therefore its center is the locus, in other words the locus is only a point.

Case (b)

As shown in Figure 2, point G is the point of intersection of the altitudes of the triangle. We denote: $\angle BAC = \alpha$, by calculating angles in the triangle we obtain $\angle BGC = 180^o - \alpha$. As the vertex A moves on the arc of the circle, the angle $\angle BA'C$ remains fixed (inscribed angle), and therefore the angle $\angle BG'C$ also remains fixed. Hence it follows that from any point G' the segment BC is observed at the same angle, and therefore it moves on a circular arc. Since the sum of the angles $\angle BA'C + \angle BG'C = 180^o$, the locus is a circle with a radius that equals the original radius, as shown in Figure 2. Since the altitudes of the triangle can intersect outside the triangle (in an obtuse triangle), it follows that a part of the locus lies outside the original circle.

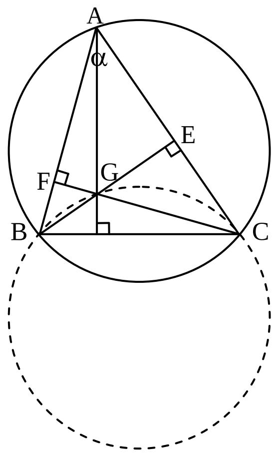

Figure 2

Case (c)

As shown in Figure 3, point G is the point of intersection of the angle bisectors. We denote $\angle BAC = \alpha$, and by calculating the angles we obtain: $\angle BGC = 90^o + \alpha/2$. In this case as well, as the point A moves along the arc of the circle, the angle $\angle BG'C$ remains constant, and therefore point G' moves on the arc of the circle in which BC is a chord. The radius of the circle which forms the locus is different than the radius of the original circle. The radii shall be equal only for the case $\alpha = 60°$. Since the point of intersection of the bisectors of the triangle is always inside the triangle, in this case the locus is inside the triangle.

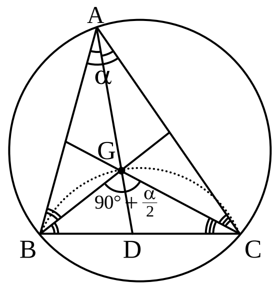

Figure 3

Case (d)

As shown in Figure 4, point G is the point of intersection of the medians of the inscribed triangle. In this case, there is no fixed relation between the angles $\angle BA'C$ and $\angle BG'C$, but there is a conserved property. As point A moves on the arc BC, the relation always takes place $A'G' = 2 \cdot G'M$, where point M is the middle of the chord BC. The relation follows from the intersection ratio of the medians in the triangle. Since point M is a fixed point when the point A moved, this point is the center of the homothety. Therefore, point G moves on a similar trajectory on which point A moves on, in other words, point G moves on a locus, which is a part of the circle that is contained entirely in the given circle.

Summary of the Cases (a)-(d)

In the cases (b)-(d) the locus of the points of intersection of the straight lines is a circle or a part of a circle, in case (a) the locus is only a point, which is in fact a degenerate circle.

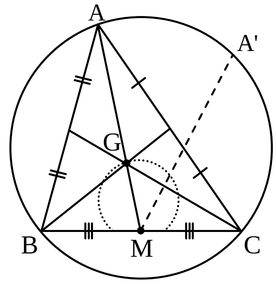

Figure 4

Applets

GeoGebra applets were prepared for cases (b)-(d), in which one can move the vertex A of the triangle and observe dynamically how the loci formed.

Note 1: When using the applets, drag only point A, on the arc of the circle, while the vertices B and C remain fixed in their places.

Note 2: The points T, L on the circle of applets allows to change the size of the circle.

Link 1: An apple for demonstrating the loci formed by the intersection point of the altitudes of the triangle inscribed in a circle (case b).
http://tube.geogebra.org/student/m892907

Link 2: An applet for demonstrating the loci formed by the intersection point of the angle bisectors of the triangle inscribed in a circle (case c).
http://tube.geogebra.org/student/m892927

Link 3: An applet for demonstrating the loci formed by the intersection point of the medians of the triangle inscribed in a circle (case d).
http://tube.geogebra.org/student/m892945

General Formulas for the Equation of the Locus of the Inscribed Center and the Point of Intersection of the Altitudes when two Vertices of the Triangle are Fixed and the Third Vertex Moves on a Given Curve

In Task 1 we considered the case where point A moves on the circular arc. Now let us consider a general case where point A moves on a given curve $y = f(x)$.

Without loss of generality, let the vertices B and C lie at $(-1,0)$ and $(1,0)$, respectively. Point A moves on some curve $y = f(x)$, and the point of intersection of the altitudes shall be $\mathrm{O}(\mathrm{X},\mathrm{Y})$, as shown in Figure 5.

The triangles ΔBOD and ΔADC are similar, therefore $\frac{\mathrm{DO}}{\mathrm{BD}} = \frac{\mathrm{DC}}{\mathrm{AD}}$. By substituting the coordinates of the vertices in the similarity ratio, one obtains:

$$\frac{\mathrm{Y}}{x+1} = \frac{1-x}{y} \quad \Rightarrow \quad \mathrm{Y} = \frac{1-x^2}{y} = \frac{1-x^2}{f(x)}$$

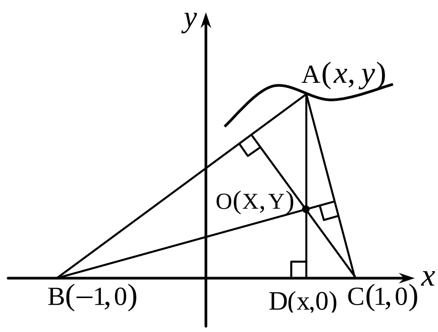

Figure 5

And since for the points A, O, D there holds $x = \mathrm{X}$, the point A moves on the curve $\mathrm{Y} = \frac{1-x^2}{f(\mathrm{X})}$.

Examples

(1) Let us assume that point A moves on the straight line $y = \mathrm{a}$ ($\mathrm{a} > 0$), a line parallel to the *x*-axis. Then point O – the intersection of the altitudes, moves on the curve $\mathrm{Y} = -\frac{x^2}{\mathrm{a}} + \frac{1}{\mathrm{a}}$ that shaped as a parabola.

(2) Let us assume that point A moves on a circle whose center lies on the *y*-axis at the point $(0, \mathrm{a})$, and whose radius is $\sqrt{\mathrm{a}^2 + 1}$ (which passes through the points B and C as shown in Figure 6).

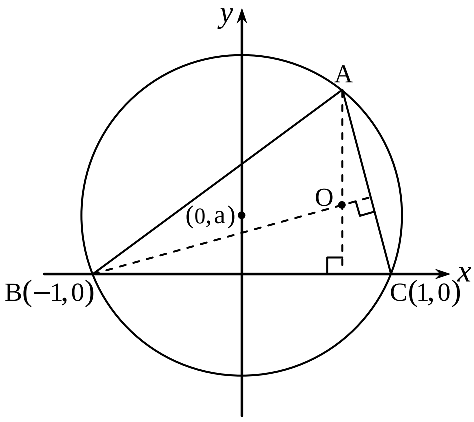

Figure 6

The equation of the circle on which point A is drawn is:

$$x^2 + (y - \mathrm{a})^2 = \mathrm{a}^2 + 1$$

or: $f(x) = y = \pm\sqrt{\mathrm{a}^2 + 1 - x^2} + \mathrm{a}$

When the equation of the curve is substituted in the relation $\mathrm{Y} = \frac{1 - x^2}{f(\mathrm{X})}$, where $x = \mathrm{X}$, we obtain that the locus on which the point O moves (the point of intersection of the altitudes) is $\mathrm{X}^2 + (\mathrm{Y} + \mathrm{a})^2 = \mathrm{a}^2 + 1$, which is a circle whose center lies at $(0, -\mathrm{a})$, and whose radius is $\sqrt{\mathrm{a}^2 + 1}$.

This conclusion can also be reached from geometric considerations. In the same manner, one can find the equation of the locus on which the point of intersection of the altitudes moves for any function $f(x)$, on which point A moves.

Task 2 – Sliding Ruler in a System of Coordinates

A ruler AB with a given length **k** slides in a system of coordinates, such that its ends A and B remains on the axes *x* and *y*, as shown in Figure 7. On what geometric trajectory do the points M and C move? (Point M is the middle of the segment, while point C lies somewhere on the segment AB)?

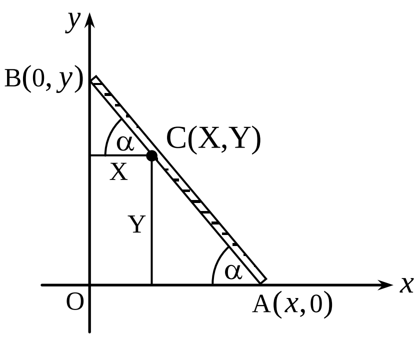

Figure 7

Proof Using Method (a) – Trigonometry

From the point $\mathrm{C}(x, y)$ we draw perpendiculars to the axes, and denote by α the acute angle between the ruler and the *x* axis. Point C divides the segment AB into two parts, the ratio of whose lengths is $\mathrm{p} = \mathrm{m} : \mathrm{n}$ ($\frac{\mathrm{BC}}{\mathrm{CA}} = \frac{\mathrm{m}}{\mathrm{n}} = \mathrm{p}$).

Based on the definitions of trigonometric functions we have $\sin\alpha = \frac{Y}{n}$, $\cos\alpha = \frac{X}{m}$. By substituting these values in the relation $\sin^2 x + \cos^2 x = 1$, we obtain $\frac{x^2}{m^2} + \frac{y^2}{n^2} = 1$, in other words point C moves on an ellipse. We have $\frac{x_c^{\,2}}{\left(\frac{kp}{1+p}\right)^2} + \frac{y_c^{\,2}}{\left(\frac{k}{1+p}\right)^2} = 1$, therefore point C moves on the perimeter of an ellipse. For the case $p = 1$ $(m = n)$, when point C coincides with the point M, the equation of the locus is $x_c^{\,2} + y_c^{\,2} = \left(\frac{k}{2}\right)^2$, in other words the locus is a canonic circle with a radius of $\frac{k}{2}$. For $p < 1$ the large axis of the ellipse and the foci are located on the *y* axis, and when $p > 1$, the large axis of the ellipse and the foci are located on the *x* axis.

Surprising note: as the ruler slides, and depending on the location of the point C, one can obtain three different loci: a circle and two ellipses, which differ from each other in the location of the foci on the axes of the system.

In terms of the location of the point C on the ruler, one can distinguish between three cases:

(1) $m = n$: Point M moves on the circle whose radius is $\frac{k}{2}$ and whose center is at the origin O. This result is not surprising because the distance OM is fixed, since $OM = \frac{k}{2}$ is a median on the hypotenuse in a right-angled triangle. The ancients knew this fact, and therefore, when they wanted to slide at beam downwards, fearing that during its slide it would move sideways, they fastened it using a strong metal chain from the center of the beam to the point O.

(2) $m < n$: Point C moves on an ellipse whose foci lie on the *y*-axis.

(3) $m > n$: Point C moves on an ellipse whose foci lie on the *x*-axis.

Proof Using Method (b) – Analytic Geometry

In accordance with the formula for division of a segment, the coordinates of point C are:

$$x_c = \frac{mx}{m+n} \;\Rightarrow\; x = \frac{m+n}{m} x_c = \left(\frac{1+p}{p}\right) x_c$$

$$y_c = \frac{ny}{m+n} \;\Rightarrow\; y = (1+p) y_c ,$$

Based on the Pythagorean Theorem, we have $x^2 + y^2 = k^2$. We substitute the values of *x* and *y* in the Pythagorean Theorem and obtain the equation of the locus on which point C moves.

Note: it is possible to prove the formation of these loci also by using mathematical tools from other fields, but the methods are long and quite complex, therefore it seems artificial to present them.

As in the subsequent tasks, we will construct an applet, which presents the formation of these loci by changing the position of point C along the ruler, while sliding it dynamically on the axes of the system.

Link 4: An applet for demonstrating the loci formed by a point on sliding ruler in a system of coordinates.
http://tube.geogebra.org/student/m892949

Task 3 – Observing Known Loci at a Right Angle

This task shall present the following cases:
(a) What is the locus from which **a circle** observed at a right angle?
(b) What is the locus from which **a canonic ellipse** observed at a right angle?
(c) What is the locus from which **a canonic hyperbola** observed at a right angle?
(d) What is the locus from which **a canonic parabola** observed at a right angle?

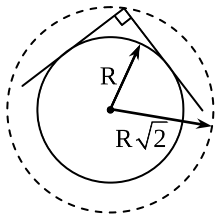

Figure 8

Case (a)

It is known that the locus from which a circle is observed at a right angle is a circle whose center coincides with the center of the original circle, and whose radius is $R\sqrt{2}$, as shown in Figure 8 (the explanation is left to the reader).

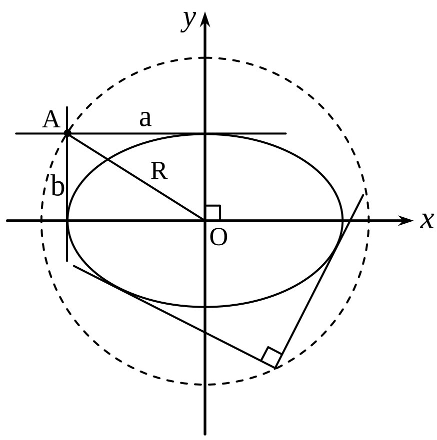

Figure 9

Case (b)

In this case the locus is also a canonic circle whose equation is $x^2 + y^2 = a^2 + b^2$, where a and b are the parameters of the ellipse. The proof of this case is simple but very long and therefore left to the reader. If we do indeed know that the locus is a canonic circle and we wish to find its radius, it is possible to draw tangents to the ellipse at the ends of the ellipse's axes. These tangents are perpendicular to each other and they intersect at the point A as shown in Figure 9. Hence the radius of the locus is $R = AO = \sqrt{a^2 + b^2}$.

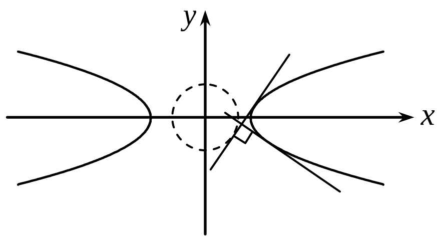

Figure 10

Case (c)

The locus in the case of the hyperbola is also a canonic circle, whose equation $x^2 + y^2 = a^2 - b^2$, where a and b are the parameters of the hyperbola (Figure 10). The proof that this is a circle is similar to the one in Case (b).

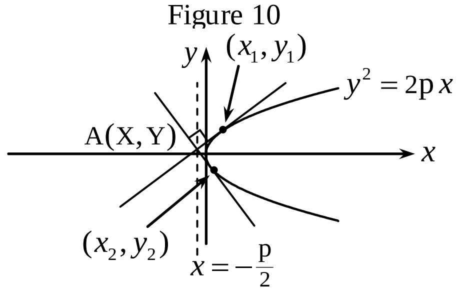

Figure 11

Case (d)

The locus from which a canonic parabola observed at a right angle is a straight line that is perpendicular to the *x*-axis (Figure 11). The result is surprising because in cases (a)-(c) the locus was a circle, and therefore it was expected that for a parabola the locus would also be a circle. This fact is a warning sign for those who wish to generalize.

Proof of Case (d)

We denote the tangency points by (x_1, y_1) and (x_2, y_2) and the point of intersection of the perpendicular tangents by $\mathrm{A}(\mathrm{X}, \mathrm{Y})$.

In accordance with the notation, the equations of the tangents shall be: $y = \frac{\mathrm{p}}{y_1}(x + x_1)$, $y = \frac{\mathrm{p}}{y_2}(x + x_2)$.

From the condition that the tangents are perpendicular, we obtain: $\frac{\mathrm{p}}{y_1} \cdot \frac{\mathrm{p}}{y_2} = -1 \;\; \Rightarrow \;\; y_1 \cdot y_2 = -\mathrm{p}^2$.

To find the point of intersection we equate the expressions for the tangents: $\frac{\mathrm{p}}{y_1}(x + x_1) = \frac{\mathrm{p}}{y_2}(x + x_2)$.

Instead of x_1 and x_2 we substitute $x_1 = \frac{{y_1}^2}{2\mathrm{p}}$, $x_2 = \frac{{y_2}^2}{2\mathrm{p}}$, and obtain that at the point of tangency there holds $x = -\frac{\mathrm{p}}{2}$. In other words, the locus from which a parabola observed at a right angle is the straight line $x = -\frac{\mathrm{p}}{2}$, the directrix of the parabola. This is a surprising result.

Task 3 can expanded by using the strategy of “what if not?” [7], which in this case leads us to ask what if the angle from which we observe the known loci is not a right angle? What if the angle is acute? What if the angle is obtuse?

The expansion of the task while using the technological tool to obtain the hypothesis allows one to include wider investigative activity and to inspect the conservation properties for other cases.

The expansion of the task gives us a wider and more profound view of the mathematical objects integrated in the task, and makes the task a powerful one.

Summary

The use of the dynamic geometric software permits immediate tracking of the formation of the locus. Such tracking visually represents the trajectory of the motion of the derived point and the trajectory of the second point whose movement is the result of dragging the first point. Such tracking simplifies the mathematical method of finding the function that describes the locus on which the second point moves.

The tasks we presented throughout the paper may serve as fertile ground for developing and extending the mathematical knowledge of pre-service teachers in the subject of loci. Tasks are relevant for integration both in the environment of teaching students/teachers and in the classroom environment. A task that is relevant for integration in the classroom environment encourages the pre-service teachers and the teachers to cooperate, to be surprised and to reflect of their mathematical knowledge while interacting with the tasks [8, 9].

References

[1] Martin, G. E. The Foundations of Geometry and the Non-Euclidean Plane. Springer-Verlag. 1975.

[2] Rutter, J. W. Geometry of Curves. Boca Raton, FL: Chapman and Hall/CRC, 2000.

[3] Chazan, D. (1993). Students' Microcomputer-Aided Explanation in Geometry. In S. I. Brown & M.I.Walter (Eds.), Problem Posing: Reflection and Application, pp. 289-299. Hillsdale, N J: Lawrence Erlbaum Associates.

[4] Dreyfus, T. and Hadas, N. (1996). Proof as Answer to the Question Why. *Zentralblatt fur Didaktik der Mathematik*, 28, 1-5.

[5] Hanna, G. (2000). Proof, Explanation and Exploration: An Overview. Educational Studies in Mthematics, 44, 5-23.

[6] Laborde, C. (2000). Dynamic Geometry Environment as a Source of Rich Learning Contexts for the Complex Activity of Providing. Educational Studies in Mathematics, 44, 151-161.

[7] Brown, S. I. and Walter, M. I. (1969). What if not? Mathematics Teaching, 46, 38-45.

[8] Zaslavsky, O. (2007). Mathematics-related tasks, teacher education, and teacher educators: The dynamics associated with tasks in mathematics teacher education. Journal for Mathematics Teacher Education, 10, 433-440.

[9] Zaslavsky, O. (2008). Meeting the challenges of mathematics teacher education through design and use of tasks that facilitate teacher learning. In B. Jaworski & T. Wood (Eds), The mathematics teacher educator as a developing professional, pp. 93-114. Sense Publishers.