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Making Use of Dynamic Software and Mathematical Tools in the Solution of Extremum Problems in Triangle Geometry

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Abstract

In triangle geometry, considerable importance is given to the study of the different properties of geometrical shapes in the context of extremum problems and inequalities. Extremum problems and inequalities occupy an important part of the mathematics study program in high school education. In tasks such as these the difficulty level is high when one does not know in advance what the expected answer is. When you know what to prove, the difficulty level is lower and most of the effort is aimed at attaining a proof of the expected answer. This can be done using dynamic geometric software (DGS). The possibility of making frequent changes to the geometrical objects, and dragging objects, contributes to the process of deducing properties, checking hypotheses and generalizing.

In the present paper we present five investigative tasks in Euclidean geometry, some are well known, while others are almost unknown and as enrichment for those interested in the subject are worthy of presentation. The tasks were given to preservice teachers of mathematics as part of an advanced course for integrating the technological computerized tool in the teaching of mathematics.

Keywords: Combining technology and mathematics, Dynamic geometric software, Extremum problems, Triangle

Introduction

In many fields people aspire to make extreme achievements. There are many examples of this. In long jump competitions the desire is to jump as far as possible, in running competitions the aspiration is to reach the end of the track in the shortest time. In industrial plants the desire is to buy the raw materials at the lowest price while selling the products at the highest price. A farmer wants to cultivate the maximum area, and diligent students want to achieve a high grade.

In different fields of mathematics there is a large variety of problems in which one seeks the condition/s for which one obtains extreme values for a certain purpose. It is important to note that in these cases extrema are characterized by transition from an increase to a decrease, or conversely, from a decrease to an increase.

Proofs of extremal problems are usually given using different methods, including: geometric, trigonometric, algebraic, and mostly using differential calculus.

In extremum problems, the problem is usually stated as: prove that for some value of one parameter a maximal or minimal value of another parameter is obtained. In other words, in the statement of the problem the condition for obtaining the extremum of the task is known, and it needs to be proven mathematically that it is correct. Sometimes the statement has another form: what is the condition for which a certain parameter reaches an extremum? It is clear that in such a statement, the difficulty is higher because one must first discover the condition.

In order to make the problem easier, as a first stage one can construct a computerized dynamic applet, which can be used to identify the hypothesis concerning the condition for which the extremum is obtained, thus reducing the level of difficulty of the task. Finding the condition for which the extremum is obtained using the dynamic software serves as "semi-proof", but does not replace the formal proof.

The Tasks of the Investigation

In Euclidean geometry one can find fascinating problems of finding the condition for obtaining a certain extreme value. It is important to start with investigative activity as a prelude to the use of mathematical tools.

In the present paper we present five investigative tasks in Euclidean geometry, some are well known and some are almost unknown and are worthy of presentation as enrichment for those interested in the subject.

The tasks were given to preservice teachers of mathematics as part of an advanced course for integrating the technological computerized tool in the teaching of mathematics.

It should be noted that in the preliminary course the students have acquired skills for constructing applets for investigating problems in geometry that are based on the use of the GeoGebra software.

As part of the present investigation, the students were required to construct some applets independently, while others were prepared for them. From didactic considerations one should be aware that sometimes more than one stage is required in order to reach the final hypothesis concerning the update of an extremum. Reaching a certain condition at the first stage constitutes preparation and guidance for the continued investigation. This is the reason why in two tasks two applets were required in each task, where the construction of the second applet was based on the insight from the investigative activity with the first applet.

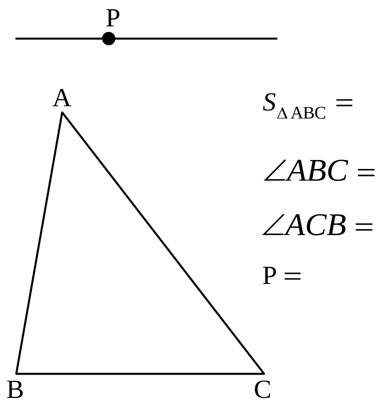
Upon completing each of the investigative tasks, a discussion with brainstorming of the methods that can be used to reach the mathematical proof, was held in the class concerning the methodical aspects of the task.

Upon completion of the computerized dynamic investigation the students were required to obtain the proof by means of independent work or by locating it in various sources, to corroborate the conclusion from the investigation, for at least two tasks.

Task A

Prove that among all the triangles with a given perimeter, the one with the largest area is an equilateral triangle. This is a classic problem ([1,2]), that is worthy of demonstrating to the students visually using the computerized tool. Figure 1 shows the visualization that appears on the computer screen.

Figure 1.



<u>Demonstration Applet</u>

Given is a slide bar that specifies the value of the perimeter of the triangle in the range of 5-25 length units.

A point P is given on the slide bar, which can be dragged in order to change the perimeter of the triangle.

The vertex B is fixed. When the vertex A is dragged the location of vertex C changes automatically, in accordance with the perimeter defined by the slide bar.

During each stage, the following values appear on the screen: The perimeter of the triangle P, the area of the triangle, the magnitudes of the angles by the vertices B and C.

When the vertex A is dragged, one can see that the maximum area is obtained for an equilateral triangle.

Link 1: Triangle with a Maximum Area.

Task B

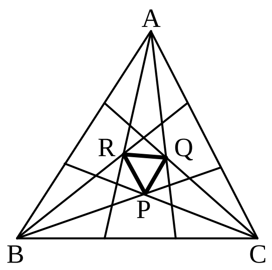
Investigating the relative area of the **Morley triangle**

In some triangles ΔABC two rays are drawn from each vertex, which divide the interior angle at this vertex into three equal parts. By connecting the points of intersection of each pair of rays that are adjacent to a particular side of the triangle, one obtains an equilateral triangle that is called the **Morley triangle**, as shown in Figure 2.

This theorem was discovered by Frank Morley in 1904 ([3]).

The investigative question is: for which triangle ΔABC will the ratio between the area of the Morley triangle and the area of ΔABC be maximal?

Figure 2.



$$\frac{S_{\Delta PQR}}{S_{\Delta ABC}} =$$

$$\angle ABC =$$

$$\angle ACB =$$

In Figure 2 obtained on the computer screen using the applet, one can see that each of the vertices of the triangle can be dragged, and that for each ΔABC one obtains a ΔPQR triangle, which is the corresponding Morley triangle. In each case one can see on the computer screen the relative area of Morley triangle and the angles of the triangle ΔABC. After dragging the vertices and tracking the relative area, one obtains that for an equilateral ΔABC, the obtained Morley triangle has the maximum relative area.

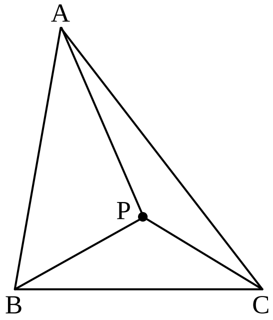
Link 2: Morley Triangle with a Maximum Relative Area.

A proof that the Morley triangle is equilateral can be found in [3,4] and the proof of the case when the Morley triangle obtains its maximum relative area can be found in [5].

Task C

Given is a triangle. Find the location of the point P inside the triangle or outside the triangle, for which the sum of the distances from that point to the vertices of the triangle is minimal (as shown in Figure 3). In other words, find the location of the point P, for which the sum of the distances $PA + PB + PC$ is minimal.

Figure 3.



This problem has been known since the 17th century. The desired point is called the Fermat-Torricelli point ([6]). Since it was first considered, it has attracted the attention of well-known mathematicians and has been given several proofs.

This problem can be regarded as an economic problem, such as: The points A, B, C are coffee shops that sell a special cake that is baked in a bakery that is located at the point P.

Where should the bakery be located so that the cakes are delivered to the coffee shops by the shortest route?

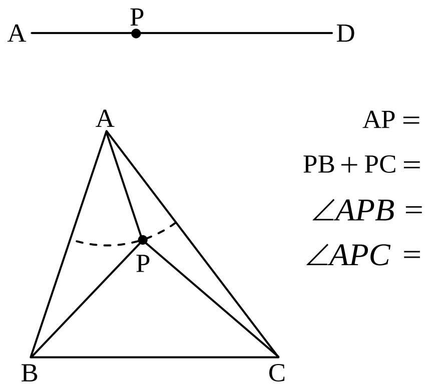
The point P for which the sum of the distances is the smallest, is the point that satisfies the condition that each of the angles $\angle APC$ and $\angle APB$ is of 120°.

This fact is true for any triangle whose largest angle is smaller than 120°. In a triangle in which the obtuse angle is larger than or equal to 120°, the location of the point P is at the vertex of the obtuse angle.

<u>The Method and Applets for Investigating the Task (Stage A)</u>

In this applet the triangle ΔABC is an acute-angled triangle, and its vertices are fixed at their locations. There is a slide bar AD, with a point P on it, which can be dragged in order to change the length AP, as shown in Figure 4. (AD is the altitude of the triangle). In this applet one can drag the point P on a circular arc with a radius of AP.

Figure 4.



In each situation, the values of AP, the sum of the lengths of the segments $\mathrm{PB}+\mathrm{PC}$, and the values of $\angle APB$ and $\angle APC$ are shown on the computer screen. The point P is dragged, and the smallest values of the sum of the lengths of the segments $\mathrm{PB}+\mathrm{PC}$ is obtained when the two angles $\angle APB$ and $\angle APC$ are equal.

[Link 3: The sum of the distances is minimal (stage a).]

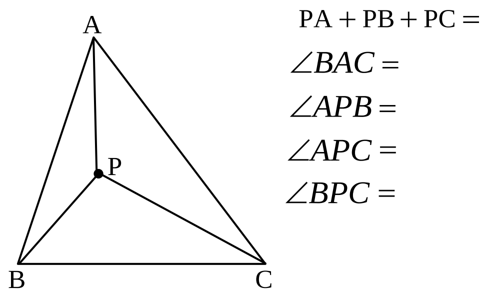
This result directs the students to investigate the task in which the point P can be dragged anywhere, not just in the circular arc with the fixed radius AP.

Applet for Stage b

In this applet, shown in Figure 5, one can drag each of the vertices of the triangle, and the point P, freely. In each situation the screen displays the sum of the lengths of the segments, $\mathrm{PA}+\mathrm{PB}+\mathrm{PC}$ and the values of the angles, $\angle BAC, \angle APB, \angle APC$ and $\angle BPC$. Point P is dragged, and it is obtained that the smallest value of the sum of the segments is when all three angles with vertex at the point P equal 120° (for the case of the triangle whose largest angle is smaller than 120°) or when point P coincides with the vertex of the obtuse angle (for the case of triangle whose largest angle is larger than or equal to 120°).

[Link 4: The sum of the distances is minimal (stage b).]

Figure 5.

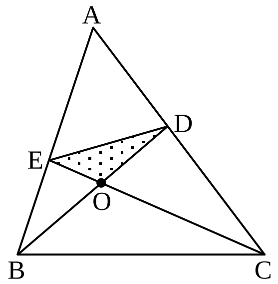


Task D

Given is a triangle ΔABC. We select a point O in the triangle. The point B is connected by a straight line to the point O, and its continuation intersects the side AC at the point D. The point C is connected by a straight line with the point O, and its continuation intersects the side AB at the point E, as shown in Figure 6 (BD and CE are cevians in the triangle ABC). The triangle ΔEOD is formed.

Prove that the maximum area of some triangle ΔEOD is obtained when the straight line ED is parallel to the side BC, and the ratio $\frac{AE}{EB}$ is equal to the golden ratio ([7].

Figure 6.



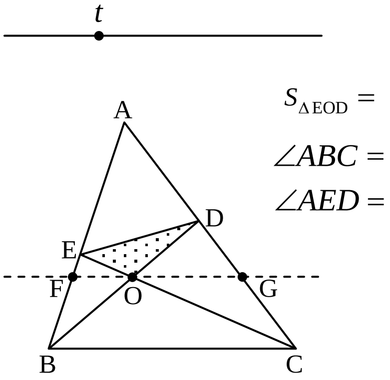
The Method and the Applets for Investigating the Task

Applet for Stage A

Link 5: The maximum area of some triangle (stage a).

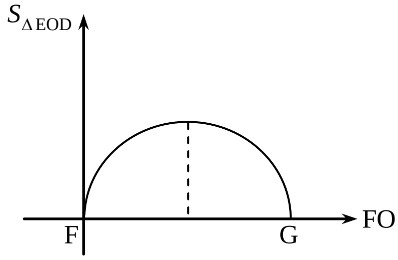
In this applet, whose visual description is shown in Figure 7, there is a slide bar *t* to change the distance between the parallel lines BC and FG. The point O lies on the parallel line, and it can be dragged on this line. When the point O is dragged on the parallel line, the area of the triangle EOD is changed. At each stage of dragging the point O, the area of the triangle and the magnitude of the angles $\angle ABC$ and $\angle AED$ appear on the screen. It can be seen that for any straight line FG that is parallel to BC, the maximum area of ΔEOD is obtained when ED is parallel to the base.

Figure 7.



The graph of variation of the area ΔEOD when the point O moves (is dragged) on the straight line FG that is parallel to BC, from one point, F, of its intersection with the side of the triangle to the other point of intersection, G, is shown in Figure 8 (the maximum area is obtained for $FO = OG$).

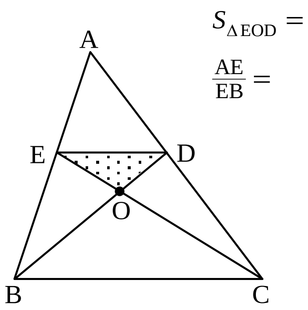
Figure 8.



Additional Stage for Investigating the Task (Stage B)

Based on the conclusion obtained from the use of the previous applet (Link 5), another applet was constructed in which the point E can be dragged on the side AB, as the segment ED remains parallel to the base BC (as shown in Figure 9).

Figure 9.



At each stage, as the point E is dragged, the area ΔEOD and the ratio $\frac{AE}{EB}$ are shown on the screen. At a certain point along the path, the maximum area of the triangle is obtained. It turns out that for this point the value of the ratio $\frac{AE}{EB}$ is $\phi = 1.618$, which is known as the Golden ratio and is the solution of the quadratic equation $x^2 - x - 1 = 0$.

Link 6: The maximum area of some triangle (stage b).

Mathematical Surprise

The maximum area of the triangle EOD is obtained at a certain point O on the straight line parallel to the base BC, where its location on the straight line depends on the distance *t*, provided that ED is parallel to the base. From Steiner's theorem for trapezoids we have: "the straight line that connects the point of intersection of the diagonals of the trapezoid with the point of intersection of the continuations of the sides of the trapezoid, bisects the basis of the trapezoid", from which it follows that the different points O which give the maximum area (in accordance with the distance of the parallel straight line from the base) lie on the median AM (M is the midpoint of the base BC). Therefore, in order to find the point O, from which the largest maximum area is obtained, it is sufficient to construct an applet in which the point O is dragged along the median AM.

Task E
Steiner's Theorem for Two Right-angled Triangles

Among other theorems of the famous geometer Jacob Steiner, one can find a theorem that concerns two right-angled triangles ΔABC and $\Delta A'B'C'$, where in the first triangle the length of the leg $AB = c$ is given, and in the second triangle the length of the leg $A'B' = c'$ is given. Also given is the sum of the lengths of the two other legs AC + A´C´ = b + b´.

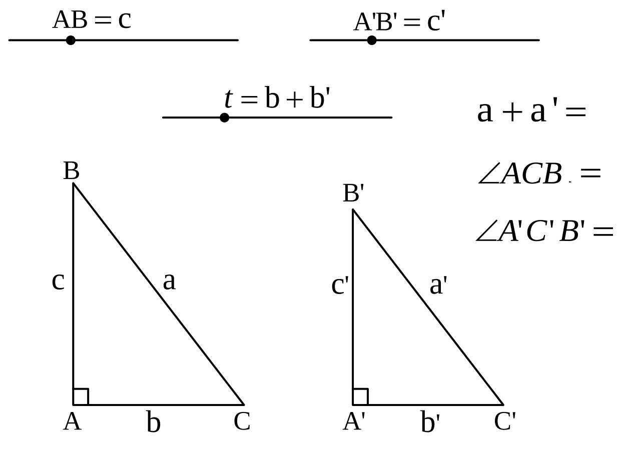
Steiner's theorem states ([8]):

The sum of the lengths of the hypotenuses $a + a'$ shall be minimal when the triangles are similar.

Demonstration Applet

The two right-angled triangles ΔABC and $\Delta A'B'C'$ are shown in Figure 10.

Figure 10.



In addition, there are three variation slide bars: a slide bar along the leg $AB = c$, a slide bar along the leg $A'B' = c'$, and a slide bar *t* for the sum of the legs $b + b'$. When the vertex C is dragged, the location of the vertex C' changes in accordance with the sum of the legs $b + b'$.

At any stage the following values appear on the screen: the sums of the lengths of the hypotenuses of the triangles $a + a'$, and the magnitudes of the angles $\angle ACB$ and $\angle A'C'B'$.

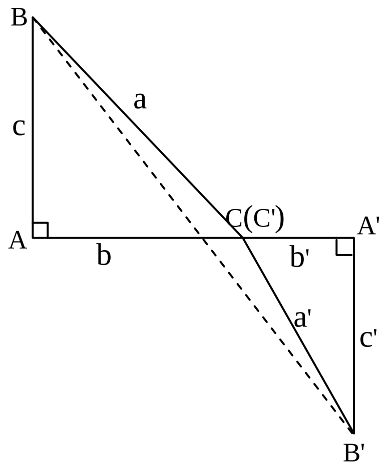
When the vertex C is dragged and one obtains the minimum value of the sum of the lengths of the hypotenuses, one can see that the two triangles have the same angles – the triangles are similar.

Link 7: The minimum value of the sum of the lengths of the hypotenuses.

The mathematical proof of the conclusion found in the dynamic investigation of this task can be simply obtained by either differential calculus or by Euclidean geometry. The proof using Euclidean geometry is very simple and is based on the theorem “the sum of the lengths of two sides of a triangle is larger than the length of the third side”.

When the triangles ΔABC and $\Delta A'B'C'$ are placed one beside the other, with the bases AC and A’C’ placed along a single line (the vertices C and C’ coincide), and the triangle $\Delta A'B'C'$ drawn inverted, as shown in Figure 11, the minimum value of the sum of the lengths of the hypotenuses $a + a'$ shall be obtained when the points B, C(C’) and B’ shall be along a single straight line, in other words – where the triangles are similar.

Figure 11.



Conclusion from Steiner’s Theorem

If the length of the leg in the first triangle is equal to the length of the leg in the second triangle, the minimum value of the sum $a + a'$ shall be obtained when the triangles are congruent.

Summary

The investigative task with the students was designed to emphasize the importance of the technological computerized tool in the teaching and learning of mathematics. During the final meeting the students expressed their satisfaction with the possibility of investigating geometric properties dynamically by changing data and observing their immediate effect, with the aim of obtaining extrema. The selected tasks contributed to increasing knowledge by becoming acquainted with new problems and theorems in mathematics.

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