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Janusz Zielinski
Associate Professor
N. Copernicus University
Poland

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Athens Institute for Education and Research
8 Valaoritou Street, Kolonaki, 10671 Athens, Greece
Tel: + 30 210 3634210 Fax: + 30 210 3634209 Email: info@atiner.gr
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Constants of Lotka-Volterra Derivations

Janusz Zielinski
Associate Professor
N. Copernicus University
Poland

Abstract

We investigate the rings of constants of Volterra and Lotka-Volterra derivations for n > 3 variables. Thus, we describe polynomial first integrals of their corresponding systems of differential equations. Such derivations play a role in population biology and in plasma physics. Moreover, they play an important part in the derivation theory itself, since they are factorizable derivations. The problem is also linked to the invariant theory. Then we describe the fields of rational constants of four-variable Volterra derivation and of generic four-variable Lotka-Volterra derivations. We also determine when an arbitrary four-variable Lotka-Volterra derivation has a nontrivial rational constant. We show how to apply these results to monomial derivations. Besides, we characterize cofactors of Darboux polynomials of arbitrary four-variable Lotka-Volterra derivations.

Keywords: Lotka-Volterra derivation, factorizable derivation, polynomial constant, rational constant, first integral.
Mathematics Subject Classification: 13N15, 12H05, 92D25, 34A34.

Introduction

The main motivations of our study are the following:

- applications of Lotka-Volterra systems in population biology, laser physics and plasma physics (see, for instance, [1] and [2]);
- Lagutinskii's procedure of association of the factorizable derivation (examples of such derivations are Lotka-Volterra derivations) with any given derivation (see, for instance, [7] and [9]);
- link to the invariant theory (for every connected algebraic subgroup G of the group $\mathrm{Gl}_n(k)$ there exists a derivation d such that $k[X]^G = k[X]^d$, see [6]).

Let us fix some notation:
k - a field of characteristic zero,
N - the set of nonnegative integers,
N_+ - the set of positive integers,
Q_+ - the set of positive rationals,
n - an integer ≥ 3,
$k[X] := k[x_1,\ldots,x_n]$,
$k(X) := k(x_1,\ldots,x_n)$.

Recall that if R is a commutative k-algebra, then a k-linear map $d : R \to R$ is called a *derivation* of R if for all a,b from R
$d(ab) = ad(b)+d(a)b$.
We call $R^d = \ker(d)$ the *ring of constants* of the derivation d. Then k is a subset of R^d and a *nontrivial* constant of d is an element of the set $R^d \setminus k$. If $f_1,\ldots, f_n$ are polynomials from $k[X]$, then there exists exactly one derivation $d : k[X] \to k[X]$ such that $d(x_1) = f_1,\ldots,d(x_n) = f_n$. A derivation $d : k[X] \to k[X]$ is said to be *factorizable* if $d(x_i) = x_i f_i$, where the polynomials f_i are of degree 1 for $i = 1,\ldots,n$. How to associate the factorizable derivation with any given derivation is presented in [9]. The construction helps to establish new facts on constants of the initial derivation (see, for instance, [7]).

There is no general effective procedure for determining $k[X]^d$ of a derivation $d : k[X] \to k[X]$, nor even deciding whether it is finitely generated (it may not be finitely generated for $n \geq 4$, see [4]). Even for a given derivation the problem may be difficult, see for instance counterexamples to Hilbert's fourteenth problem (all of them are of the form $k[X]^d$, however it took more than a half century to find at least one of them, for more details we refer the reader to [4] and [6]) or Jouanolou derivations (where the rings of constants are trivial, see [6]).

We now formulate the problem. Let $C_1,\ldots,C_n$ belong to k. Throughout the rest of this paper $d : k[X] \to k[X]$ is a derivation of the form
$d(x_i) = x_i(x_{i-1}-C_i x_{i+1})$
for $i = 1,\ldots,n$ (we adhere to the convention that $x_{n+1} = x_1$ and $x_0 = x_n$). Our task is to determine $k[X]^d$.

The problem has only partial solutions:

- J. Moulin Ollagnier, A. Nowicki [5]: for $n = 3$ and arbitrary parameters C_i;
- P. Hegedűs [3]: for an arbitrary n but all C_i equal to 1;
- J. Zieliński [14]: for $n = 4$ and arbitrary parameters C_i.

If $C_i = 1$ for all i, then we call d a *Volterra derivation*. It has some symmetries. Let D_{2n} denote the dihedral group on n vertices. For f from $k[X]^d$ and σ from D_{2n} we have f^σ in $k[X]^d$, that is, the ring $k[X]^d$ is D_{2n}-invariant.

A stronger fact is true. If n is even, then let $G < D_{2n}$ denote the stabiliser of $\{1, 3, 5,\ldots, n\text{-}1\}$. If n is odd, then let $G = D_{2n}$.

Corollary 1. ([3]). *The constants of d are G-invariant, that is* $k[X]^d$ *is a subset of* $k[X]^G$.

We call a polynomial g from $k[X]$ *strict* if it is homogeneous and not divisible by the variables $x_1,\ldots,x_n$. For $\alpha = (\alpha_1,\ldots,\alpha_n)$ from N^n, we denote by X^α the monomial $x_1^{\alpha 1}\ldots x_n^{\alpha n}$. Every nonzero homogeneous polynomial f from $k[X]$ has the unique representation $f = X^\alpha g$, where X^α is a monomial and g is strict.

A nonzero polynomial f is said to be a *Darboux polynomial* (or an *integral element*) of a derivation $d : k[X]\rightarrow k[X]$ if $d(f) = \Lambda f$ for some Λ from $k[X]$. We will call Λ a *cofactor* of f. Since d is a homogeneous derivation of degree 1, the cofactor of each homogeneous polynomial is a linear form. Denote by $k[X]_{(m)}$ the homogeneous component of $k[X]$ of degree m.

Lemma 1. ([15]). *Let* $n = 4$. *Let g from* $k[X]_{(m)}$ *be a Darboux polynomial of d with the cofactor* $\lambda_1 x_1 +\ldots+\lambda_4 x_4$. *Let i belong to* $\{1,2,3,4\}$. *If g is not divisible by* x_i, *then* λ_{i+1} *belong to N. More precisely, if* $g(x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_4) = x_{i+2}^{\beta i+2}G$ *and G is not divisible by* x_{i+2}, *then* $\lambda_{i+1} = \beta_{i+2}$ *and* $\lambda_{i+3} = -C_{i+2}\lambda_{i+1}$.

Corollary 2. ([15]). *Let* $n = 4$. *If g from* $k[X]$ *is a strict Darboux polynomial, then its cofactor is a linear form with coefficients in N.*

Lemma 2. ([15]). *Let* $n = 4$. *If* $d(f) = 0$ *and* $f = X^\alpha g$, *where g is strict, then* $d(X^\alpha) = 0$ *and* $d(g) = 0$.

Let $k[X]^d_{(m)} = k[X]_{(m)} \cap k[X]^d$. Since d is homogeneous, $k[X]^d$ is a direct sum of $k[X]^d_{(m)}$ for m from N and we need only to determine the homogeneous constants.

Restrictions of *n*-Variable Polynomials

In this section we investigate the restrictions of polynomials to polynomial rings in smaller number of variables. The considerations are also valid for a number of variables greater than 4, so they may be useful in some further studies as well.

Let φ belong to $k[X]$ and $1 \leq q \leq n$. Then for every subset A of the set $\{1,\ldots,n\}$ we denote by φ^A the sum of terms of φ that depend on variables with indices in A, that is, $\varphi^A = \varphi|_{xj=0 \text{ for } j \text{ not in } A}$.

By the *support* of $\alpha = (\alpha_1,\ldots,\alpha_n)$ from N^n we mean the set $\text{supp}(\alpha) = \{i : \alpha_i \neq 0\}$ and we denote by $|\alpha|$ the sum $\alpha_1+\ldots+\alpha_n$.

Lemma 3. ([15]). *If A is a subset of $\{1,\ldots,n\}$, then for every φ from $k[X]^d_{(m)}$ we have* $d(\varphi^A)^A = 0$.

We noticed that in the proofs it is much more convenient to deal with polynomials φ such that $d(\varphi^A)^A = 0$ for a given set A, than with constants themselves.

Lemma 4. ([15]). *Let φ belong to $k[X]_{(m)}$ and $A = \{i, i+1\}$ be a subset of $\{1,\ldots,n\}$. If $d(\varphi^A)^A = 0$, then $\varphi^A = a(x_i+C_ix_{i+1})^m$ for some a from k.*

Lemma 5. ([15]). *Let $n \geq 4$, φ belong to $k[X]_{(m)}$ and i is an element of the set $\{1,\ldots,n\}$. Let C_i not belong to Q_+ and $A = \{i, i+1, i+2\}$. If $d(\varphi^A)^A = 0$, then φ^A belongs to $k[x_i+C_ix_{i+1}+C_iC_{i+1}x_{i+2}]$.*

Lemma 6. ([14]). *Let $n \geq 3$ and $m \geq 1$. If φ belongs to $k[X]^d_{(m)}$, then*
$\varphi = a(x_1+C_1x_2+C_1C_2x_3+\ldots+C_1\ldots C_{n-1}x_n)^m+\sum b_\alpha X_\alpha$,
where the latter sum is taken over all $|\alpha| = m$ such that $\#\mathrm{supp}(\alpha) \geq 3$ or $\#\mathrm{supp}(\alpha) = 2$ and these two nonzero exponents are not of consecutive variables (in the cyclic sense), a, b_α are elements of k for all α. Moreover, if $C_1\ldots C_n \neq 1$, then $a = 0$.

Lemma 7. ([14]). *Let $n \geq 4$, φ belong to $k[X]_{(m)}$, i belong to $\{1,\ldots,n\}$ and $A = \{i, i+1, i+2\}$. Let C_i belong to Q_+ and $C_i = p/q$, where p, q belong to N_+ and* $\gcd(p,q) = 1$. If $d(\varphi^A)^A = 0$, then φ^A belongs to $k[x_i+C_ix_{i+1}+C_iC_{i+1}x_{i+2}, x_i^qx_{i+2}^p]$.

Four-Variable Lotka-Volterra Derivations

In this section we assume that $n = 4$.

Lemma 8. ([8]). *$k[X]^d$ contains a nontrivial monomial constant if and only if at least one of the following two conditions is fulfilled*:
(1) C_1, C_3 *belong to Q_+ and* $C_1C_3 = 1$,
(2) C_2, C_4 *belong to Q_+ and* $C_2C_4 = 1$.

Consider the three sentences:
s_1 : $C_1C_2C_3C_4 = 1$.
s_2 : C_1, C_3 belong to Q_+ and $C_1C_3 = 1$.
s_3 : C_2, C_4 belong to Q_+ and $C_2C_4 = 1$.

In case s_2 let $C_1 = p/q$, where p, q belong to N_+ and $\gcd(p,q) = 1$. In case s_3 let $C_2 = r/t$, where r, t belong to N_+ and $\gcd(r,t) = 1$.

Denote by $\neg s_i$ the negation of the sentence s_i.

Lemma 9. ([14]). *Let φ belong to $k[X]^d_{(m)}$. If $\neg s_2$, then $\varphi^{\{1,2,3\}}$ belongs to $k[x_1+C_1x_2+C_1C_2x_3]$ and $\varphi^{\{3,4,1\}}$ belongs to $k[x_3+C_3x_4+C_3C_4x_1]$.*
Let $n = 4$ and $d : k[X]\rightarrow k[X]$ be a derivation of the form
$d = \sum^4_{i=1} x_i(x_{i-1}-C_ix_{i+1})\partial/\partial x_i$,

where C_1,C_2,C_3,C_4 are elements of k. In the notation s_1, s_2, s_3, p, q, r, t introduced above, we have the following main theorem. Note that the conjunction of the sentences s_2 and s_3 implies s_1. This means that we have only seven cases to consider, depending on the truth values of the sentences s_1, s_2, s_3.

<u>Theorem 1. ([14]).</u> *The ring of constants of d is always finitely generated over k with at most three generators. In each case it is a polynomial ring, more precisely*:
(1) *if* $s_1 \wedge \neg s_2 \wedge \neg s_3$, *then* $k[X]^d = k[x_1+C_1x_2+C_1C_2x_3+C_1C_2C_3x_4]$,
(2) *if* $\neg s_1 \wedge \neg s_2 \wedge \neg s_3$, *then* $k[X]^d = k$,
(3) *if* $\neg s_1 \wedge \neg s_2 \wedge s_3$, *then* $k[X]^d = k[x^t_2x^r_4]$,
(4) *if* $\neg s_1 \wedge s_2 \wedge \neg s_3$, *then* $k[X]^d = k[x^q_1x^p_3]$,
(5) *if* $s_1 \wedge \neg s_2 \wedge s_3$, *then* $k[X]^d = k[x_1+C_1x_2+C_1C_2x_3+C_1C_2C_3x_4,\ x^t_2x^r_4]$,
(6) *if* $s_1 \wedge s_2 \wedge \neg s_3$, *then* $k[X]^d = k[x_1+C_1x_2+C_1C_2x_3+C_1C_2C_3x_4,\ x^q_1x^p_3]$,
(7) *if* $s_2 \wedge s_3$, *then* $k[X]^d = k[x_1+C_1x_2+C_1C_2x_3+C_1C_2C_3x_4,\ x^q_1x^p_3,\ x^t_2x^r_4]$.

As a conclusion, we determine the existence of nontrivial polynomial constants.

<u>Corollary 3. ([14]).</u> *If* $n = 4$, *then* $k[X]^d$ *contain a nontrivial polynomial constant if and only if at least one of the following three conditions is fulfilled*:
(1) $C_1C_2C_3C_4 = 1$,
(2) C_1, C_3 *belong to* Q_+ *and* $C_1C_3 = 1$,
(3) C_2, C_4 *belong to* Q_+ *and* $C_2C_4 = 1$.

Rational Constants

For any derivation δ: $k[X]\rightarrow k[X]$ there exists exactly one derivation $\bar{d}$: $k(X)\rightarrow k(X)$ such that $\bar{d}|_{k[X]} = \delta$. By a rational constant of the derivation δ : $k[X]\rightarrow k[X]$ we mean the constant of its corresponding derivation $\bar{d}$: $k(X)\rightarrow k(X)$. The rational constants of δ form a field. For simplicity, we often write δ instead of $\bar{d}$.

<u>Proposition 1. ([6]).</u> *Let* $\delta : k[X]\rightarrow k[X]$ *be a derivation and let f and g be nonzero relatively prime polynomials from* $k[X]$. *Then* $\delta(f/g) = 0$ *if and only if f and g are Darboux polynomials of* δ *with the same cofactor.*

<u>Proposition 2. ([6]).</u> *Let* δ *be a homogeneous derivation of* $k[X]$ *and let f from* $k[X]$ *be a Darboux polynomial of* δ *with the cofactor* Λ *from* $k[X]$. *Then* Λ *is homogeneous and each homogeneous component of f is also a Darboux polynomial of* δ *with the same cofactor* Λ.

<u>Proposition 3. ([6]).</u> *Let* δ *be a derivation of* $k[X]$. *If f belonging to* $k[X]$ *is a Darboux polynomial of* δ, *then all factors of f are Darboux polynomials of* δ.

Theorem 2. ([10]). *If d is the four-variable Volterra derivation, then* $k(X)^d = k(x_1+x_2+x_3+x_4,\ x_1x_3,\ x_2x_4)$.

Denote by D_0 the field of fractions of an integral domain D.

Corollary 4. *If d is the four-variable Volterra derivation, then* $k(X)^d=(k[X]^d)_0$.

From now on, $n = 4$. For C_1,C_2,C_3,C_4 from k consider the sentences:
$\check{s}_2$: C_1, C_3 belong to Q and $C_1C_3 = 1$.
$\check{s}_3$: C_2, C_4 belong to Q and $C_2C_4 = 1$.

In case $\check{s}_2$ let $C_1 = p/q$, where p, q belong to Z, $q \neq 0$ and $\gcd(p,q) = 1$. In case $\check{s}_3$ let $C_2 = r/t$, where r, t belong to Z, $t \neq 0$ and $\gcd(r,t) = 1$. Note that these presentations of C_i are unique up to sign. Sentences s_1, s_2 and s_3 are as in Section 3.

Theorem 3. ([11]). *Let* $d : k(X)\rightarrow k(X)$ *be a four-variable Lotka-Volterra derivation with parameters* C_1,C_2,C_3,C_4 *from k. Then*:

(1) *if* $s_1 \wedge\neg\check{s}_2 \wedge\neg\check{s}_3$, *then* $k(X)^d = k(x_1+C_1x_2+C_1C_2x_3+C_1C_2C_3x_4)$,
(2) *if* $\neg s_1 \wedge\neg\check{s}_2 \wedge\neg\check{s}_3$, *then* $k(X)^d = k$,
(3) *if* $\neg s_1 \wedge\neg\check{s}_2 \wedge s_3$, *then* $k(X)^d = k(x^t{}_2x^r{}_4)$,
(4) *if* $\neg s_1 \wedge s_2 \wedge\neg\check{s}_3$, *then* $k(X)^d = k(x^q{}_1x^p{}_3)$,
(5) *if* $s_1 \wedge\neg\check{s}_2 \wedge s_3$, *then* $k(X)^d = k(x_1+C_1x_2+C_1C_2x_3+C_1C_2C_3x_4,\ x^t{}_2x^r{}_4)$,
(6) *if* $s_1 \wedge s_2 \wedge\neg\check{s}_3$, *then* $k(X)^d = k(x_1+C_1x_2+C_1C_2x_3+C_1C_2C_3x_4,\ x^q{}_1x^p{}_3)$,
(7) *if* $s_2 \wedge s_3$, *then* $k(X)^d = k(x_1+C_1x_2+C_1C_2x_3+C_1C_2C_3x_4,\ x^q{}_1x^p{}_3,\ x^t{}_2x^r{}_4)$.

Corollary 5. ([11]). *If d is a four-variable Lotka-Volterra derivation, then* $k(X)^d$ *contains a nontrivial rational constant if and only if at least one of the following three conditions is fulfilled*:

(1) $C_1C_2C_3C_4 = 1$,
(2) C_1, C_3 *belong to Q and* $C_1C_3 = 1$,
(3) C_2, C_4 *belong to Q and* $C_2C_4 = 1$.

Note that if d is as in Theorem 3, then the field of rational constants equals the field of fractions of the ring of polynomial constants (similarly to Corollary 4). Which is not true in general:

Example 1. Let $k = R$ or $k = C$. Let $d : k[X]\rightarrow k[X]$ be a derivation defined by
$d(x_i) = x_i(x_{i-1}+x_{i+1})$, for $i = 1, 3$,
$d(x_i) = x_i(x_{i-1}-\sqrt{2}x_{i+1})$, for $i = 2, 4$.
By Theorem 1, $k[X]^d = k$. Nevertheless, $x_{1/}x_3$ belongs to $k(X)^d$.

We say that a derivation $\delta : k(X)\rightarrow k(X)$ is *monomial* if $\delta(x_i) = x_1^{\beta i1}\ldots x_n^{\beta in}$ for $i = 1,\ldots,n$, where each βij is an integer. Theorem 4 is an application of Lotka-Volterra derivations to monomial derivations.

Theorem 4. ([13]). *Let* $s_1,\dots,s_4$ *belong to* N_+, *where* $(s_1, s_3) \neq (1, 1)$ *and* $(s_2, s_4) \neq (1, 1)$. *Let* $D : k(X) \rightarrow k(X)$ *be a derivation of the form*

$$D(x_i) = x^{si-1+1}_{i-1}\, x^{si+1}_{i}\, x^{si+2}_{i+2}$$

for $i = 1,\dots,4$ (*in the cyclic sense*). *Then* $k(X)^D = k$.

References

Almeida, M. A., Magalhães, M. E., and Moreira, I. C. 1995. Lie symmetries and invariants of the Lotka-Volterra system. *J. Math. Phys.* 36, 1854-1867.

Cairó, L., and Llibre, J. 2000. Darboux integrability for 3D Lotka-Volterra systems. *J. Phys. A* 33, 2395-2406.

Hegedűs, P. 2012. The constants of the Volterra derivation. *Cent. Eur. J. Math.* 10, 969-973.

Kuroda, S. 2005. Fields defined by locally nilpotent derivations and monomials. *J. Algebra* 293, 395-406.

Moulin Ollagnier, J., and Nowicki, A. 1999. Polynomial algebra of constants of the Lotka-Volterra system. *Colloq. Math.* 81, 263-270.

Nowicki, A. 1994. *Polynomial derivations and their rings of constants.* N. Copernicus University Press, Toruń.

Nowicki, A., and Zieliński, J. 2006. Rational constants of monomial derivations. *J. Algebra* 302, 387-418.

Ossowski, P., and Zieliński, J. 2010. Polynomial algebra of constants of the four variable Lotka-Volterra system. *Colloq. Math.* 120, 299-309.

Zieliński, J. 2007. Factorizable derivations and ideals of relations. *Comm. Algebra* 35, 983-997.

Zieliński, J. 2014. Fields of rational constants of LV derivations. Preprint.

Zieliński, J. 2014. The field of rational constants of the Volterra derivation. *Proc. Est. Acad. Sci.* 63, 133-135.

Zieliński, J. 2011. The five-variable Volterra system. *Cent. Eur. J. Math.* 9, 888-896.

Zieliński, J. 2013. Rational constants of generic LV derivations and of monomial derivations. *Bull. Pol. Acad. Sci. Math.* 61, 201-208.

Zieliński, J. 2013. Rings of constants of four-variable Lotka-Volterra systems. *Cent. Eur. J. Math.* 11, 1923-1931.

Zieliński, J., and Ossowski, P. 2013. Rings of constants of generic 4D Lotka-Volterra systems. *Czechoslovak Math. J.* 63, 529-538.