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Abstract

In recent years, the subject of different representations of the same field in mathematics has seen considerable development. This includes different representations of certain theorems, such as the very important Ceva's theorem. This development is a complement of the subject of fusion of fields in mathematics, where the solution of a particular problem is achieved using mathematical tools from the same field or from different fields, thus exhibiting the beauty of mathematics as a tree built from interwoven branches and sub-branches. The present paper describes different representations of Ceva's theorem and shows how the use of these theorems brings about surprising and wide-ranging results in the study of the geometric properties of a triangle.

Keywords: Ceva's theorem, cevians, triangle geometry.

Introduction

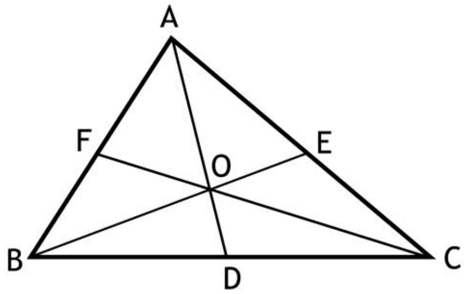
Ceva's theorem was originally published by the Italian mathematician Giovani Ceva in 1678 (see for example [1,2,3]). This theorem plays an important role in Euclidean geometry, and especially in the geometry of the triangle. Since Ceva's theorem concerns segments in the triangle which connect its vertices with the opposite sides, these segments are named cevians after Ceva. Ceva's theorem provides a sufficient and necessary condition for the three cevians to meet at the same point (such cevians are called concurrent cevians). Ceva's theorem is related and constitutes a link to many other theorems in geometry. This theorem has many different proofs and several different representations which result in surprising and interesting results. Some of them are given in the paper.

Different Representations of CEVA'S Theorem

The Classical Representation

Three concurrent cevians AD, BE, CF which intersect at the same point O are given in the triangle ABC (see Figure 1).

Figure 1.



Hence,

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=1 \text{ (I)}$$

One of the known proofs of Ceva's theorem is by means of Menelaus' theorem.

Menelaus' theorem states that for the triangle ABD that is intersected by a straight line passing through the points C, O, F, there holds:

$$\frac{BC}{DC}\cdot\frac{DO}{OA}\cdot\frac{AF}{FB}=1, \qquad \text{therefore } \frac{AO}{DO}=\frac{BC}{DC}\cdot\frac{AF}{FB} \text{ (1).}$$

By using Menelaus' theorem again for the triangle ADC that is intersected by a straight line passing through the points E, O, B, we obtain:

$$\frac{BC}{BD}\cdot\frac{DO}{OA}\cdot\frac{AE}{EC}=1, \qquad \text{therefore } \frac{AO}{DO}=\frac{BC}{BD}\cdot\frac{AE}{EC} \text{ (2)}$$

From relations (1) and (2) we obtain that $\frac{BC \cdot AF}{DC \cdot FB} = \frac{BC \cdot AE}{BD \cdot EC}$, and therefore $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$.

Note 1:
As we have shown, Ceva's theorem follows simply and directly from Menelaus' theorem. It is therefore surprising that some 1700 years have passed between discoveries of these two theorems.

Representation by Van Aubel's Theorem

From relations (1) and (2) it can be easily obtained that $\frac{AO}{DO} \cdot \left(\frac{DC}{BC} + \frac{BD}{BC} \right) = \frac{AF}{FB} + \frac{AE}{EC}$. Since $\frac{DC}{BC} + \frac{BD}{BC} = \frac{DC+BD}{BC} = \frac{BC}{BC} = 1$, we obtain that $\frac{AO}{DO} = \frac{AF}{FB} + \frac{AE}{EC}$ (II).

This is a famous theorem by Van Aubel (see for example [4]), that claims that if three cevians AD, BE, CF intersect at the point O, then (II) holds. The Van Aubel theorem is an alternative representation of Ceva's theorem, where the classical representation has a multiplicative nature and the representation by Van Aubel's theorem has an additive nature.

Representation by Ratios of Areas

We give a proof to Ceva's theorem by considering the areas of the triangles AOB, AOC, BOC, which we shall denote by S_{AOB}, S_{AOC}, and S_{BOC} respectively).

The following relations hold for the areas:

$$\frac{S_{AOB}}{S_{AOC}} = \frac{S_{BOD}}{S_{DOC}} = \frac{BD}{DC} \quad (3)$$

$$\frac{S_{BOC}}{S_{AOB}} = \frac{S_{OEC}}{S_{OEA}} = \frac{CE}{AE} \quad (4)$$

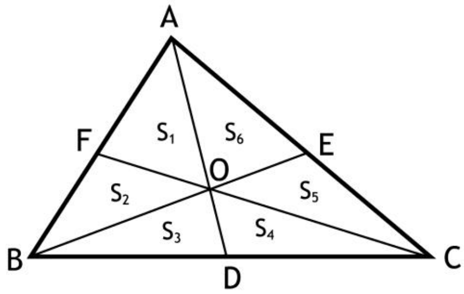
$$\frac{S_{AOC}}{S_{BOC}} = \frac{S_{AOF}}{S_{BOF}} = \frac{AF}{FB} \quad (5)$$

We multiply the relations (3), (4) and (5) and obtain that $1 = \frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB}$.

Hence it follows that the three cevians passing through the point O divide the triangle into 6 triangles with areas of S_1, S_2, …, S_6 (Figure 2), so that $\frac{S_1 \cdot S_3 \cdot S_5}{S_2 \cdot S_4 \cdot S_6} = 1$ (III).

This last relation is an alternative representation of Ceva's theorem using area ratios.

Figure 2.



Trigonometric Representations

The First Trigonometric Representation

It can be seen that

$$\frac{S_{ABD}}{S_{ADC}} = \frac{AB \sin A_1}{AC \sin A_2} = \frac{BD}{DC} \quad (6)$$

$$\frac{S_{BCE}}{S_{BAE}} = \frac{BC \sin B_1}{AB \sin B_2} = \frac{CE}{EA} \quad (7)$$

$$\frac{S_{CAF}}{S_{CBF}} = \frac{AC \sin C_1}{BC \sin C_2} = \frac{AF}{FB}. \quad (8)$$

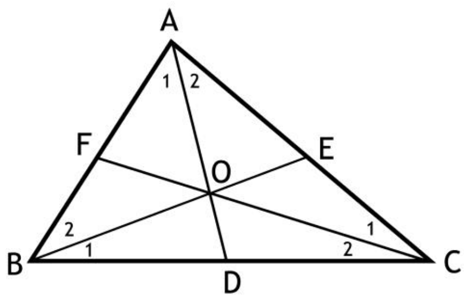
We multiply the relations (6), (7) and (8) and obtain:

$$\frac{\sin A_1}{\sin B_2} \cdot \frac{\sin B_1}{\sin C_2} \cdot \frac{\sin C_1}{\sin A_2} = \frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} \quad (9).$$

By using the Law of Sines in the triangles AOB, BOC and COA, it follows that $\frac{\sin A_1}{\sin B_2} = \frac{BO}{AO}$, $\frac{\sin B_1}{\sin C_2} = \frac{CO}{BO}$, $\frac{\sin C_1}{\sin A_2} = \frac{AO}{CO}$. It therefore also follows that the left-hand side in relation (9) is equal to 1, and therefore we conclude that the three cevians in the triangle ABC that pass through the point O divide the angles of the triangle into pairs of angles A_1, A_2, C_1, C_2, B_1, B_2 (Figure 3), such that $\sin A_{1.} \cdot \sin B_1 \cdot \sin C_1 = \sin A_{2.} \cdot \sin B_2 \cdot \sin C_2$ (IV).

This is the alternative representation of Ceva's theorem using trigonometry.

Figure 3.



The Second Trigonometric Representation

The area ratios (Figure 4) can be written as:

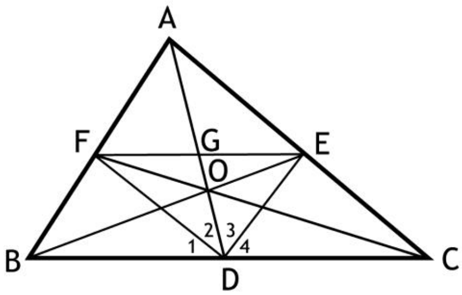
$$\frac{S_{ADF}}{S_{BDF}} = \frac{AD\sin D_2}{BD\sin D_1} = \frac{AF}{BF} \quad (10)$$

$$\frac{S_{CDE}}{S_{ADE}} = \frac{DC\sin D_4}{AD\sin D_3} = \frac{CE}{AE} \quad (11)$$

From relations (10) and (11) we obtain that $\frac{\sin D_2 \cdot \sin D_4}{\sin D_1 \cdot \sin D_3} = \frac{AF}{BF} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA}$ (12).

From relation (12) one can formulate the second trigonometric representation of Ceva's theorem as follows: the three cevians in the triangle ABC that pass through the point O divide the angle D (Figure 4) into four angles D_1, D_2, D_3, D_4, such that: $\frac{\sin D_2 \cdot \sin D_4}{\sin D_1 \cdot \sin D_3} = 1$ (V).

Figure 4.



Surprizing Applications of Representations of Ceva's Theorem

Proposition 1

Three concurrent cevians divide the given triangle (whose area is S) into 6 triangles. It is clear that the area of at least one of them does not exceed S/6. What is surprising is that there is at least another triangle whose area also does not exceed S/6.

Proof:

$$\frac{S}{6} = \frac{\sum_{i=1}^{6} S_i}{6} \geq \sqrt[6]{\prod_{i=1}^{6} S_i} \text{ . From (III), } \sqrt[6]{\prod_{i=1}^{6} S_i} = \sqrt[3]{S_1 \cdot S_3 \cdot S_5} = \sqrt[3]{S_2 \cdot S_4 \cdot S_6} \text{ .}$$

Therefore $\frac{S}{6} \geq Min(S_1, S_3, S_5)$ and $\frac{S}{6} \geq Min(S_2, S_4, S_6)$.

Proposition 2

Three concurrent cevians divide the angles of the triangle to produce 6 angles. It is clear that at least one of them does not exceed 30°, but it turns out that there is at least another angle whose value also does not exceed 30°.

Proof:

$$\sin A_{1.} \cdot \sin A_2 = 0.5(\cos(A_1 - A_2) - \cos A) = 0.5(\cos(A_1 - A_2) - 1 + 2\sin^2(A/2)) \le \sin^2(A/2)$$

Hence,

$$\sin A_{1.} \cdot \sin B_1 \cdot \sin C_1 \cdot \sin A_{2.} \cdot \sin B_2 \cdot \sin C_2 \le \sin^2(A/2)\sin^2(B/2)\sin^2(C/2).$$

Therefore, from the representation (IV), we have: $\sin A_{1.} \cdot \sin B_1 \cdot \sin C_1 \le \sin(A/2)\sin(B/2)\sin(C/2)$.

It is known that the following inequality holds for any triangle: $\sin(A/2)\sin(B/2)\sin(C/2) \le 1/8$ (see for example [5]), therefore: $\sin A_{1.} \cdot \sin B_1 \cdot \sin C_1 \le 1/8$ and $Min(\sin A_{1.}, \sin B_1, \sin C_1) \le 1/2$.

This suggests that at least one of the angles A_1, B_1, C_1 does not exceed 30°. In the same manner we obtain that at least one of the angles A_2, B_2, C_2 also does not exceed 30°.

Proposition 3

Consider the triangle DEF. It turns out that its area is at least 4 times smaller than the area of the triangle ABC.

We denote: BD/DC = α; CE/EA = β; AF/FB = γ.

From the representation (I) we have $\alpha\beta\gamma = 1$, and there holds: $1+\alpha \ge 2\sqrt{\alpha}$, $1+\beta \ge 2\sqrt{\beta}$, $1+\gamma \ge 2\sqrt{\gamma}$, therefore $(1+\alpha)(1+\beta)(1+\gamma) \ge 8\sqrt{\alpha\beta\gamma} = 8$ (13)

From Routh's theorem [3, p.382; 6] there holds $S_{DEF} = \dfrac{S_{ABC}(1+\alpha\beta\gamma)}{(1+\alpha)(1+\beta)(1+\gamma)}$, and from this and from relation (13) we obtain that $S_{DEF} \le \dfrac{S_{ABC}}{4}$.

Proposition 4

Representation (II) yields an interesting inequality: $\dfrac{AO}{OD} + \dfrac{BO}{OE} + \dfrac{CO}{OF} \ge 6$.

Proof:

From the representation (II) there holds: $\dfrac{AO}{DO} = \dfrac{AF}{FB} + \dfrac{AE}{EC}$, in a similar manner we have $\dfrac{BO}{OE} = \dfrac{BD}{DC} + \dfrac{BF}{FA}$ and $\dfrac{CO}{OF} = \dfrac{CE}{EA} + \dfrac{CD}{DB}$, and by adding the three expressions together with obtain:

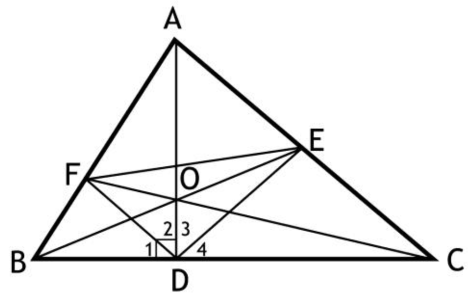
$$\frac{AO}{OD} + \frac{BO}{OE} + \frac{CO}{OF} = \left(\frac{AF}{FB} + \frac{BF}{AF}\right) + \left(\frac{BD}{DC} + \frac{DC}{BD}\right) + \left(\frac{CE}{EA} + \frac{EA}{CE}\right) \ge 2 + 2 + 2 = 6$$

where equality holds when the three cevians are the medians of the triangle.

Proposition 5

Use of the representation (V) immediately suggests that if one cevian, for example AD, is the altitude in the triangle ABC, then for any two other cevians that meet on AD, there holds that the angle D_1 is equal to the angle D_4 (see Figure 5).

Figure 5.



Indeed, if $\angle BDA=90^o$ then $\sin D_1 = \cos D_2$ and $\sin D_4 = \cos D_3$. From the representation (V) we have $\frac{\sin D_2 \cdot \sin D_4}{\sin D_1 \cdot \sin D_3} = 1$ therefore $\frac{\sin D_2 \cdot \cos D_3}{\cos D_2 \cdot \sin D_3} = 1$, and therefore $tgD_2 = tgD_3$ and $\angle D_2 = \angle D_3$, $\angle D_1 = \angle D_4$.

We note that this suggests that in the triangle known as the orthic triangle that is formed by the endpoints of the altitudes of the triangle ABC, these altitudes bisect its angles.

Proposition 6

From representation (III) we have that $\frac{S_1 \cdot S_3 \cdot S_5}{S_2 \cdot S_4 \cdot S_6} = 1$. Hence follows an interesting fact that in order to calculate the areas S_1, S_2,…, S_6, it is enough to know only three of the areas. This proof requires complex algebra transformations and it will therefore not be shown.

We shall demonstrate this fact using a numeric example. Assuming that it is given that $S_1 = 3, S_2 = 7, S_3 = 4$, we calculate S_4, S_5, and S_6.

To this end we denote $S_4 = x, S_5 = y, S_6 = z$. Since there holds $\frac{S_1 + S_2}{S_3} = \frac{S_6 + S_5}{S_4} = \frac{AO}{OD}$, we have $\frac{10}{4} = \frac{y+z}{x}$, or $y + z = \frac{5x}{2}$ (14).

In a similar manner, $\frac{S_4 + S_3}{S_2} = \frac{S_6 + S_5}{S_1} = \frac{CO}{OF}$, and hence $\frac{y+z}{3} = \frac{x+4}{7}$ and $y + z = \frac{3(x+4)}{7}$ (15).

From relations (14) and (15) we find that $x = \frac{24}{29}$, and therefore $y + z = \frac{60}{29}$.

From representation (III) one can write $3 \cdot 4 \cdot y = 7 \cdot \frac{24}{29} \cdot z$, and therefore $z = \frac{60}{43}, y = \frac{840}{1247}$.

Proposition 7

For the three concurrent cevians there holds: $\frac{AO}{OD} = 2 \cdot \frac{AG}{GD}$ (see Figure 4).

Proof:

We denote: BD/DC = α, CE/EA = β, AF/FB = γ.

Then $\frac{AG}{GD} = \frac{S_{AFE}}{S_{DFE}} = \frac{\frac{S_{AFE}}{S_{ABC}}}{\frac{S_{DFE}}{S_{ABC}}} = \frac{\frac{AF \cdot AE}{AB \cdot AC}}{\frac{S_{DFE}}{S_{ABC}}} = \frac{\frac{\gamma}{\gamma+1} \cdot \frac{1}{\beta+1}}{\frac{S_{DFE}}{S_{ABC}}}$.

Since $\frac{S_{DFE}}{S_{ABC}} = \frac{2}{(1+\alpha)(1+\beta)(1+\lambda)}$, we have $\frac{AG}{GD} = \frac{\gamma(\alpha+1)}{2}$ (16).

From representation (II) we have $\frac{AO}{DO} = \gamma + \frac{1}{\beta}$, and therefore from (I), $\frac{AO}{DO} = \frac{\gamma\beta+1}{\beta} = \frac{\frac{1}{\alpha}+1}{\beta} = \frac{1+\alpha}{\alpha\beta} = \gamma(1+\alpha)$ (17).

By comparing relations (16) and (17), we have $\frac{AO}{OD} = 2 \cdot \frac{AG}{GD}$.

<u>Note 2:</u>

From proposition 7 it can be easily deduced that if three points A, O, D lie on the same straight line, so that $\frac{AO}{OD} = k$, then one can construct a point G that satisfies $\frac{AG}{GD} = \frac{k}{2}$ <u>using a straightedge only</u>.

And there are of course other applications of Ceva's theorem which result in interesting and surprising facts. We only presented some of them.

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