Athens Institute for Education and Research
ATINER

ATINER's Conference Paper Series
EMS2014-1079

The Use of Computational Mathematics in Teaching of Ramsey Numbers: A Case Study at Kean University

Kung-Kuen Tse
Assistant Professor
Kean University
USA

An Introduction to ATINER's Conference Paper Series

ATINER started to publish this conference papers series in 2012. It includes only the papers submitted for publication after they were presented at one of the conferences organized by our Institute every year. The papers published in the series have not been refereed and are published as they were submitted by the author. The series serves two purposes. First, we want to disseminate the information as fast as possible. Second, by doing so, the authors can receive comments useful to revise their papers before they are considered for publication in one of ATINER's books, following our standard procedures of a blind review.

Dr. Gregory T. Papanikos
President
Athens Institute for Education and Research

This paper should be cited as follows:

Tse, K.K., (2014) **"The Use of Computational Mathematics in Teaching of Ramsey Numbers: A Case Study at Kean University",** Athens: ATINER'S Conference Paper Series, No: **EMS2014-1079.**

Athens Institute for Education and Research
8 Valaoritou Street, Kolonaki, 10671 Athens, Greece
Tel: + 30 210 3634210 Fax: + 30 210 3634209 Email: info@atiner.gr
URL: www.atiner.gr
URL Conference Papers Series: www.atiner.gr/papers.htm
Printed in Athens, Greece by the Athens Institute for Education and Research.
ISSN: **2241-2891**
29/07/2014

The Use of Computational Mathematics in Teaching of Ramsey Numbers: A Case Study at Kean University

Kung-Kuen Tse
Assistant Professor
Kean University
USA

Abstract

Ramsey number $R(m,n)$ is the smallest integer N such that, in any red-blue 2-coloring of the edges of a N-vertex complete graph K_N, there must exist an all-red K_m or an all-blue K_n. Ramsey numbers are very difficult to determine. There are two approaches to determine Ramsey numbers, theoretical and computational. We carried out these two approaches to teach the students Ramsey numbers at Kean University to see which approach is better. Based on the data collected, we found that the students have a better understanding of the subject using the computational approach.

.

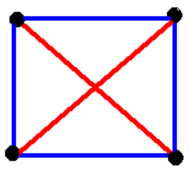
Keywords: Computational mathematics, Ramsey numbers, graph theory, graph coloring, computer programming.

Introduction

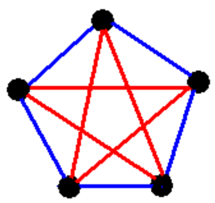
Frank P. Ramsey introduced Ramsey theory in 1930 to study the conditions under which order must appear in certain combinatorial arrangements. It can be viewed as a far-reaching generalization of the elementary and well-known pigeon hole principle. Ramsey Numbers are the center of Ramsey theory. These numbers quantify some of the general existential theorems in Ramsey theory. In this paper, we introduce and study Ramsey numbers in a way that demonstrates some advantages of the computational approach compared to the conventional abstract proof method. Finally, we report on the success of an experimental course based on this approach at Kean University.

What are Ramsey Numbers?

Suppose there are 4 points on a piece of paper and you have two pens: one blue and one red. The task is to draw a single edge between any two of these points using one of these pens. Is there a way to connect the points so that the final picture does not contain a blue triangle or a red triangle? The answer is yes: connect the 4 points to form a blue square and connect the diagonals with red.



Now repeat the same experiment with 5 points: is there a way to connect the points so that the final picture does not contain a blue triangle or a red triangle? The answer is again yes: connect the 5 points to form a blue pentagon and draw any other edges in red.



Finally, repeat the same experiment with 6 vertices. This time, I claim that any such attempt will fail: the final picture will always contain either a blue triangle or a red triangle. This is the first result of Ramsey theory: starting at *n* points, $n \geq 6$, no matter how these points are connected with blue or red edges. Fortunately, there is another approach: utilizing computers, there will always be a blue triangle or a red triangle as long as any two vertices are connected by an edge.

Denote K_n the *complete graph* on *n* vertices, which is defined as the graph having any two of its vertices connected by an edge. Using this concept, we define the *Ramsey number* $R(m,n)$ as the minimal number of vertices that will result in a blue graph K_m or a red graph K_n, when each two of these vertices are connected either by a blue or by a red edge.

The example we considered above corresponds to the case $R(3,3)$ and our first result states that

$$R(\Delta,\Delta)=R(3,3)=6$$

How to Teach Ramsey Numbers?

Traditionally, Ramsey numbers are taught and calculated by abstract arguments. When this approach is used, students learn graph theory from the perspective of combinatorial mathematics, with long computations justified by elaborate mathematical reasoning. Fortunately, there is another approach, based on utilizing computers, which is the focus of this paper. We find this approach to be more appealing than the conventional one, because most students are intimidated by abstract mathematics.

Finding Ramsey numbers is rather computation-intense and repetitive, which makes computers an indispensable and logical tool for the task. In addition to finding Ramsey numbers themselves, this approach provides a good motivation for students in their learning of algorithmic thinking and computer programming. In the experimental course at Kean, this subject was used as a gateway to graph theory, combinatorics, algorithms and Unix. The course prerequisites included only basic computer programming and data structures.

Benefits

Some novel computational techniques for calculating (generalized) Ramsey numbers were introduced (Babak, Radziszowski and Tse 2004, Radziszowski and Tse 2002). When working on those research projects, the authors realized that the subject was very fitting for an introductory course in computational methods. The following properties make it particularly suitable for that task.

Our Experience

Some novel computational techniques for calculating (generalized) Ramsey numbers were introduced (Babak, Radziszowski and Tse 2004, Radziszowski and Tse 2002). When working on those research projects, the authors realized that the subject was very fitting for an introductory course in computational methods. The following properties make it particularly suitable for that task.

Ramsey number is very visual, clear and easy to understand; the description of the problem itself doesn't require any sophisticated mathematics and can be described in plain language that even a kid can understand. This was confirmed in the pilot course: using their intuition, students were able to imply the meaning of a slightly more general notation. For example, the class quickly realized that $R(\Delta,\Delta,\Delta)$ was about finding monochromatic triangles when using 3 color pens instead of 2 color pens to connect the dots and $R(\Delta,\Diamond)$ was about the possibility of finding a blue triangle or a red square after connecting dots with blue and red pens.

As the classical Ramsey numbers $R(m,n)$ are extremely difficult to compute, only a few are computed exactly and many are still undetermined to this day. (Radziszowski 2014) has compiled all the results on Ramsey numbers known at that time. When in 1995, McKay and Radziszowski computed $R(4,5)=25$ (McKay and Radziszowski 1995), the result was impressive enough to make it into the New York Times Science section.)

While the subject of generalized Ramsey numbers is equally far from its completion, many special cases yield themselves more easily to computational approach, enabling students to contribute to science while still learning the basics of programming. The possibility of being the first person in the world to discover an unknown Ramsey number could give tremendous motivation to a student entering this field.

Ramsey numbers are closely related to graph theory. The generalized Ramsey number is to replace the K_n by all kinds of graphs: n-cycle graph C_n, path graph P_n, book graph B_n, wheel graph W_n and star graph. Each type of graphs has its unique properties, students need to adjust the algorithms accordingly, so they need to examine the graphs carefully and find ways to speed up the computations.

When students study the run time of algorithms, they learn all kinds of big O-symbols in theory, but they seldom need to run programs for a long period of time to see the difference among different algorithms. Therefore Ramsey numbers give a very good opportunity to appreciate the need for fast algorithms. For small Ramsey numbers (Ramsey numbers on graphs with small number of vertices), one computer or a few computers are sufficient; but for large Ramsey numbers, since the complexity increased exponentially, one has to utilize hundreds or even thousands of computers to process a huge amount of graphs. Gradually, students realize the need to learn about computer cluster and parallel programming language, and discover the need and usefulness of supercomputers. The language MPI was used in the course for implementing multi-processor algorithms.

Ramsey numbers are a good way to discover many useful tools available in Unix platform. For example, sometimes students need to use the "nice" command so that their processes don't use too much resources of the system.

Occasionally, they also write Unix scripts or use pipes on command line to automate a multi-stage operation.

Our Experience

The pilot course was run at Kean University for one semester. There were 7 sophomores in the class, 4 majored in mathematics and 3 majored in computer science. The key was to lure students into learning new ideas every step of the way: At the beginning, we asked them to use the one point extension method (adding one vertex at a time) to generate graphs and determine which one is a Ramsey graph.

They quickly found that the method was too slow and infeasible. Then we introduced them to more advanced algorithms and cluster programming techniques running on all the 100+ computers in the lab. The students were required to implement their own programs, run them, collect the data and compare the results to ensure the validity of their code and correctness of the results. At the end of the semester, even though we did not discover any new Ramsey numbers, we were able to verify and duplicate some of the results listed in Radziszowski 2014. Students were more communicative, showed more initiative, and were more positive about the experience.

Comparisons of the Two Approaches

Before the pilot course, we taught Ramsey numbers for two semesters using the conventional approach. The assignments were different. For example, in the conventional approach, one exercise is to use counting argument to prove that $R(K_3,C_4)=6$. In the computational approach, a typical exercise is to implement the gluing algorithm to compute $R(C_4,C_4)$. We intentionally made the final exam questions identical because it gave a clear indication on how well the students understood the material using the two different approaches. The exam, which was 3 hour long, consisted of 5 problems. One of the questions in the final exam was to demonstrate that $R(K_3,C_5)=9$. Here is the performance on the final exam between the two approaches:

Table 1. *Data of the Two Approaches of Teaching Ramsey Numbers*

	Conventional 1st semester	**Conventional 2nd semester**	**Computational Approach**
Students attended	15	13	16
Students' grade	90,87,86,86	88,87,82,81	95,93,92,90
	80,77,77,75	76,75,71,70	88,87,87,81
	70,65,62,61	66,65,62,54	79,77,75,74
	60,51,43	40	74,73,71,64

The *p*-value for the first and third column of data is 0.029784 and the *p*-value for the second and third column of data is 0.024248. It shows a significant improvement on the student's understanding of the subject by using the computational approach.

Conclusions

In the pilot course, the students set out to conquer some of the undetermined Ramsey numbers. While they did not discover any new numbers, at the end, they learned graph theory, algorithms, parallel programming and — perhaps most importantly — got their first experience on how to conduct research. The data provides a convincing evidence of improved learning. We think this approach is far better than studying Ramsey numbers the more traditional abstract way.

References

Babak, A. Radziszowski, S. and Tse, K.-K. 2004. Computational of the Ramsey Numbers $R(B_3,K_5)$. *Bulletin of the Institute of Combinatorics and its Applications*. 41, 71-76.

McKay, B. and Radziszowski, S. 1995. R(4,5). *Journal of Graph Theory*. 19, 309-322.

Ramsey, F. P. 1930. On a problem of formal Logic. *Proceedings of London Mathematical Society*. Series 2, 20, 264-286.

Radziszowski, S. 2014. Small Ramsey Numbers. *Electronic Journal of Combinatorics* http://www.combinatorics.org.

Radziszowki, S. and Tse, K.-K. 2002. A Computational Approach for the Ramsey Numbers $R(C_4,K_n)$. *The Journal of Combinatorial Mathematics and Combinatorial Computing*, 42, 195-207.