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Abstract

Unified hybrid censoring is a mixture of generalized Type-I and Type-II hybrid censoring schemes. This article presents the statistical inferences on Weibull parameters when the data are unified hybrid censored. It is observed that the maximum likelihood estimators (MLEs) cannot be obtained in closed form. We propose to use the EM algorithm to compute the maximum likelihood estimators. We obtain the observed Fisher information matrix using the missing information principle and it can be used for constructing the asymptotic confidence intervals. We also obtain the Bayes estimates of the unknown parameters under the assumption of independence using the Gibbs sampling procedure. Simulations are performed to compare the performances of the different methods and for illustrative purposes we have analyzed one data set.

Keywords: Bayes estimators; EM algorithm; Fisher information matrix; Gibbs sampling; Maximum likelihood estimators; Unified hybrid censoring.

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Introduction

Consider a life-testing experiment in which n identical units are placed on a life-test. Let $X_{1:n} \le X_{2:n} \le ... \le X_{n:n}$ denote the corresponding lifetimes from an Weibull distribution with probability density function $f(x)$ and cumulative distribution function $F(x)$.

Generalized Type-I and Type-II hybrid censoring schemes (HCS) was first introduced by Chandrasekar et al., (2004). A mixture of generalized Type-I and Type-II HCS is known as the unified HCS and it can be described as follows: Suppose n identical units are put to test under the same environmental conditions and the lifetime of each unit is independent and identically distribution (i.i.d) random variables. Fix $k, r \varepsilon \{1,2,...,n\}$ and $T_1 < T_2 \varepsilon (0,\infty)$ such that $k < r$. If k-th failure occurs before time T_1, the experiment terminate at $\min\{\max\{X_{r:n}, T_1\}, T_2\}$; if the k-th failure occurs between T_1 and T_2, the experiment terminate at $\min\{X_{r:n}, T_2\}$ and if the k-th failure occurs after time T_2, then the experiment terminate at $X_{k:n}$. Under this censoring scheme, we can guarantee that the experiment would be completed at most in time T_2 with at least k failure and if not, we can guarantee exactly k failures. Balakrishnan et al. (2008), first introduced the unified HCS and analyzed the data under the assumption of exponential lifetime distribution of the experimental units. They also obtained exact confidence intervals for the mean of the exponential distribution under the unified HCS.

In this paper, we consider the analysis of the unified HCS lifetime data when the lifetime of each experimental unit follows two parameters Weibull distribution. Weibull distribution is one of the most common distribution which is used to analyze several lifetime data. The aim of this paper is two fold. First we consider the point and interval estimates of the unknown parameters, based on the frequentist approach. It is observed that the MLEs can be obtained by solving two non-linear equation, but they can not be obtained in closed form. Although, the standard Newton-Raphson algorithm can be employed to solve the non-linear equation, but unfortunately it does not converge all the time even from good starting values. We propose to use the EM algorithm to compute the MLEs. Using the missing information principle we calculate the observed Fisher information matrix, which can be used for constructing the asymptotic confidence intervals of the unknown parameters. The second aim of this paper is to consider the Bayesian inference for the unknown parameters when the data are unified HCS. The Bayes estimates can not be obtained in closed form. Using the Gibbs sampling procedure we obtain the Bayes estimates and also the highest posterior density (HPD) credible intervals under the assumptions of independent of both the shape and scale parameters. Simulations are performed to compare the performances of the different methods.

Model Description

Suppose the lifetime random variable X has a Weibull distribution with the shape and scale parameters as α and λ respectively, probability density function (pdf) of X is;

$$f_X(x;\alpha,\lambda) = \frac{\alpha}{\lambda}\left(\frac{x}{\lambda}\right)^{\alpha-1} e^{-\left(\frac{x}{\lambda}\right)^{\alpha}}; \quad x > 0, \tag{1}$$

where $\alpha > 0$, $\lambda > 0$ are the natural parameters space. If the random variable X has the density function (1), then $Y = \ln X$ has the extreme value distribution with pdf;

$$f_Y(y;\mu,\sigma) = \frac{1}{\sigma} e^{\frac{y-\mu}{\sigma} - e^{\frac{y-\mu}{\sigma}}}; \quad -\infty < y < \infty, \tag{2}$$

where $\mu = \ln \lambda$, $\sigma = \dfrac{1}{\alpha}$. Models (1) and (2) are equivalent models in the sense, the procedure developed under one model can be easily used for the other model. Although, they are equivalent models, sometimes it is easier to work with the model (2) than (1), because in the model (2), the two parameters μ and σ appear as location and scale parameters, respectively. In fact, it is observed that the approximate MLEs can be obtained quite easily using model (2) than model (1).

Now we describe the data available under the unified HCS. Note that, under the unified HCS, it is assumed that $r, k\varepsilon\{1,2,...,n\}$, $T_1 < T_2\varepsilon(0,\infty)$ such that $k < r$ are known in advance. Thus, under this censoring scheme we have six cases:

(1) $0 < X_{k:n} < X_{r:n} < T_1 < T_2$ $\quad the\, experiment terminate\, at\, T_1,$

(2) $0 < X_{k:n} < T_1 < X_{r:n} > T_2$ $\quad terminate\, at\, X_{r:n},$

(3) $0 < X_{k:n} < T_1 < T_2 < X_{r:n}$ $\quad terminate\, at\, T_2,$

(4) $0 < T_1 < X_{k:n} < X_{r:n} < T_2$ $\quad terminate\, at\, X_{r:n},$

(5) $0 < T_1 < X_{k:n} < T_2 < X_{r:n}$ $\quad terminate\, at\, T_2,$

(6) $0 < T_1 < T_2 < X_{k:n} < X_{r:n}$ $\quad terminate\, at\, X_{k:n}.$

Maximum Likelihood Estimators

In this section we provide the MLEs of the unknown parameters based on the observation given in Section 2. First we write down the likelihood function in six cases separately. Let D_j denote the number of failures until time T_j, $j = 1,2$. Then, the likelihood function of unified HCS is as follow:

$$L(\sigma,\mu)\propto\begin{cases}(\frac{1}{\sigma})^{D}e^{\sum_{i=1}^{D}\frac{x_{i:n}-\mu}{\sigma}}e^{-[\sum_{i=1}^{D}e^{\frac{x_{i:n}-\mu}{\sigma}}+(n-D)e^{\frac{T_1-\mu}{\sigma}}]} & D_1=D_2=D=r,\dots,n,\\ (\frac{1}{\sigma})^{r}e^{\sum_{i=1}^{r}\frac{x_{i:n}-\mu}{\sigma}}e^{-[\sum_{i=1}^{r}e^{\frac{x_{i:n}-\mu}{\sigma}}+(n-r)e^{\frac{x_{r:n}-\mu}{\sigma}}]} & D_1=k,\dots,r-1;D_2=r,\\ (\frac{1}{\sigma})^{D_2}e^{\sum_{i=1}^{D_2}\frac{x_{i:n}-\mu}{\sigma}}e^{-[\sum_{i=1}^{D_2}e^{\frac{x_{i:n}-\mu}{\sigma}}+(n-D_2)e^{\frac{T_2-\mu}{\sigma}}]} & D_1=k,\dots,r-1;\\ & D_2=k,\dots,r-1;D_1\le D_2,\\ (\frac{1}{\sigma})^{r}e^{\sum_{i=1}^{r}\frac{x_{i:n}-\mu}{\sigma}}e^{-[\sum_{i=1}^{r}e^{\frac{x_{i:n}-\mu}{\sigma}}+(n-r)e^{\frac{x_{r:n}-\mu}{\sigma}}]} & D_1=0,\dots,k-1;D_2=r,\\ (\frac{1}{\sigma})^{D_2}e^{\sum_{i=1}^{D_2}\frac{x_{i:n}-\mu}{\sigma}}e^{-[\sum_{i=1}^{D_2}e^{\frac{x_{i:n}-\mu}{\sigma}}+(n-D_2)e^{\frac{T_2-\mu}{\sigma}}]} & D_1=0,\dots,k-1;\\ & D_2=k,\dots,r-1,\\ (\frac{1}{\sigma})^{k}e^{\sum_{i=1}^{k}\frac{x_{i:n}-\mu}{\sigma}}e^{-[\sum_{i=1}^{k}e^{\frac{x_{i:n}-\mu}{\sigma}}+(n-k)e^{\frac{x_{k:n}-\mu}{\sigma}}]} & D_2=0,\dots,k-1.\end{cases}$$

Note the likelihood function in six cases can be combined as follow

$$L(\sigma,\mu)\propto(\frac{1}{\sigma})^{d}e^{\sum_{i=1}^{d}\frac{x_{i:n}-\mu}{\sigma}}e^{-(\sum_{i=1}^{d}e^{\frac{x_{i:n}-\mu}{\sigma}}+(n-d)e^{\frac{c-\mu}{\sigma}})}, \qquad (3)$$

here

$$\begin{aligned}&\text{for case } I: && d=D, c=T_1,\\ &\text{for cases } II, IV: && d=r, c=x_{r:n},\\ &\text{for cases } III, V: && d=D_2, c=T_2,\\ &\text{for case } VI: && d=k, c=x_{k:n}.\end{aligned}$$

Taking the logarithm of (3), we obtain

$$l(\sigma,\mu)=-d\ln\sigma+\sum_{i=1}^{d}\frac{x_{i:n}-\mu}{\sigma}-\sum_{i=1}^{d}e^{\frac{x_{i:n}-\mu}{\sigma}}-(n-d)e^{\frac{c-\mu}{\sigma}}, \qquad (4)$$

and derivatives with respect to σ and μ of (4) are

$$\begin{aligned}\frac{\partial l}{\partial\sigma}&=-\frac{d}{\sigma}-\sum_{i=1}^{d}\frac{x_{i:n}-\mu}{\sigma^2}+\sum_{i=1}^{d}\frac{x_{i:n}-\mu}{\sigma^2}e^{\frac{x_{i:n}-\mu}{\sigma}}+(n-d)\frac{c-\mu}{\sigma^2}e^{\frac{c-\mu}{\sigma}}=0,\\ \frac{\partial l}{\partial\mu}&=-\frac{d}{\sigma}+\frac{1}{\sigma}\sum_{i=1}^{d}e^{\frac{x_{i:n}-\mu}{\sigma}}+\frac{(n-d)}{\sigma}e^{\frac{c-\mu}{\sigma}}=0.\end{aligned} \qquad (5)$$

It is clear that the likelihood equations in (5) are implicit, we need some numerical techniques to solve the simultaneous equations. We suggest to use the EM algorithm to compute the MLEs and it is described below. The EM algorithm, originally proposed by Dempster et al. (1977), is a very powerful tools in handling the incomplete data problem. We treat this problem as a missing value problem similarly as in Ng et al. (2002). Let us denote the observed and the censored data by $X = (X_{1:n}, X_{2:n}, ..., X_{d:n})$ and $Z = (Z_1, Z_2, ..., Z_{n-d})$ respectively. Here for a given d and $Z_1, Z_2, ..., Z_{n-d}$ are not observable. The censored data vector Z can be thought of as missing data. The combination of $W = (X, Z)$ forms the complete data set. The log-likelihood function based on the complete log-lifetime W is

$$l(\sigma, \mu; w) \quad = -n \ln \sigma + \sum_{i=1}^{d} \frac{x_{i:n} - \mu}{\sigma} - \sum_{i=1}^{n-d} \frac{z_i - \mu}{\sigma} - \sum_{i=1}^{d} e^{\frac{x_{i:n} - \mu}{\sigma}} - \sum_{i=1}^{n-d} e^{\frac{z_i - \mu}{\sigma}}. \qquad (6)$$

In E-step one needs to compute the pseudo log-likelihood function as $E(l(\sigma, \mu; w) \mid X)$,

$$\begin{aligned} E(l(\sigma, \mu; w) \mid X) \quad &= -n \ln \sigma + \sum_{i=1}^{d} \frac{x_{i:n} - \mu}{\sigma} - \sum_{i=1}^{n-d} E\left(\frac{z_i - \mu}{\sigma} \mid z_i > c\right) \\ &- \sum_{i=1}^{d} e^{\frac{x_{i:n} - \mu}{\sigma}} - \sum_{i=1}^{n-d} E\left(e^{\frac{z_i - \mu}{\sigma}} \mid z_i > c\right). \end{aligned} \qquad (7)$$

The conditional distribution of Z_i given $\underline{X} = \underline{x}$ is a truncated extreme value distribution with left truncation at c have pdf (Ng et al., 2002)

$$f_Z(z_i \mid Z_i > c, \sigma, \mu) = \frac{e^{e^{\xi}}}{\sigma} e^{\frac{z_i - \mu}{\sigma} - e^{\frac{z_i - \mu}{\sigma}}}; \quad z_i > c, \ \xi = \frac{c - \mu}{\sigma}. \qquad (8)$$

The ML estimation of the parameters based on the complete data from the extreme value distribution can not be solved explicitly. However, this problem has been well studied (Lawless, 1982). Thus, in the M-step of the (h+1)-th iteration of the EM algorithm, the value of $\sigma_{(h+1)}$ is first obtained by solving the equation

$$\begin{aligned} \sigma_{(h+1)} \quad &= \frac{\sum_{i=1}^{d} x_{i:n} e^{(x_{i:n}/\sigma_{(h+1)})} + (n-d) E[Z_i e^{Z_i/\sigma} \mid z_i > c, \sigma_{(h)}, \mu_{(h)}]}{\sum_{i=1}^{d} e^{(x_{i:n}/\sigma_{(h+1)})} + (n-d) E[e^{Z_i/\sigma} \mid z_i > c, \sigma_{(h)}, \mu_{(h)}]} \\ &- \frac{1}{n} \left(\sum_{i=1}^{d} x_{i:n} + (n-d) E(Z_i \mid z_i > c, \sigma_{(h)}, \mu_{(h)}) \right), \end{aligned} \qquad (9)$$

and then obtain $\mu_{(h+1)}$ by

$$\mu_{(h+1)} = \sigma_{(h+1)} \ln \left(\frac{1}{n} \left(\sum_{i=1}^{d} e^{(x_{i:n}/\hat{\sigma}_{(h+1)})} + (n-d) E(e^{Z_i/\sigma} \mid z_i > c, \sigma_{(h)}, \mu_{(h)}) \right) \right). \qquad (10)$$

Fisher Information Matrices

In this section we present the observed Fisher information matrix obtained using the missing value principle of Louis (1982). The observed Fisher information matrix can be used to construct the asymptotic confidence intervals. The idea of missing information principle is as follows;

Observed information = Complete information - Missing information.

Let us use the following notation; $\theta = (\sigma, \mu)$, $X = \text{the observed data}$, $W = \text{the complete data}$, $I_W(\theta) = \text{the observed information}$, $I_{W|X} = \text{the missing information}$, then $I_X(\theta) = I_W(\theta) - I_{W|X}(\theta)$. From the classical results on the extreme value distribution, the complete data information matrix is (Stephens, 1977)

$$I_W(\theta) = \frac{n}{\sigma^2}\begin{pmatrix} 1 & 1-\gamma \\ 1-\gamma & c^2 \end{pmatrix}, \quad (11)$$

where $\gamma = 0.57721566\,5$ is the Euler's constant and c^2 is $\pi^2/6 + (1-\gamma)^2 = 1.82368066\,1$. The Fisher information matrix of the censored observation can be written as

$$I_{W|X}(\theta) = -(n-d)E_{Z|X}\left[\frac{\partial^2 \ln f_Z(z\,|\,X,\theta)}{\partial \theta^2}\right].$$

The asymptotic variance covariance matrix of $\hat{\theta}$ can be obtained by inverting $I_X(\hat{\theta})$.

Bayes Estimates

In this section, we consider the Bayes estimations of the unknown parameters and also constructions of the credible intervals. We re-parametrize the model as follows $\theta = \frac{1}{\lambda^{\alpha}}$. Based on the new parametrization, we consider the Bayes estimates of α and θ. Unfortunately, when both the parameters are unknown then there is not exist any natural conjugate priors. Similarly as in Berger and Sun (1993), it is assumed that θ has a gamma prior, $\Gamma(a,b)$, for $a, b > 0$, i.e.

$$\pi_1(\theta) \propto \theta^{a-1} e^{-b\theta}; \quad \theta > 0. \quad (12)$$

No specific form of priors $\pi_2(\alpha)$ on α is assumed here. It is only assumed that the support of $\pi_2(\alpha)$ is $(0, \infty)$ and it is independent of θ. Based on the above prior assumptions, the joint posterior density function α and θ is

$$\pi(\alpha, \theta\,|\,\text{data}) \propto \alpha^d \theta^{a+d-1} (\Pi_{i=1}^d x_{i:n})^{\alpha-1} e^{-\theta(\sum_{i=1}^{d} x_{i:n}^{\alpha} + (n-d)c^{\alpha} + b)} \pi_2(\alpha). \quad (13)$$

Therefore, the Bayes estimate of any function of α and θ, say $\nu(\alpha,\theta)$ under the squared error loss function is

$$\hat{\nu}_B = E_{\alpha,\theta|\mathrm{d}ata}(\nu(\alpha,\theta)) = \frac{\int_0^\infty\int_0^\infty \nu(\alpha,\theta) l(\mathrm{d}ata \mid \alpha,\theta)\pi_1(\theta)\pi_2(\alpha)d\alpha\, d\theta}{\int_0^\infty\int_0^\infty l(\mathrm{d}ata \mid \alpha,\theta)\pi_1(\theta)\pi_2(\alpha)d\alpha\, d\theta}. \quad (14)$$

It is not possible to compute (15) analytically in this case. We will provide the Gibbs sampling procedures to compute the point estimate of any function of α and θ. To perform the Gibbs sampling procedure, we further assume that $\pi_2(\alpha)$ is log- concave. It may be mentioned that the well known distributions like Weibull and gamma have log-concave density functions if the corresponding shape parameters are greater than or equal to one, whereas normal and log-normal have always log-concave density function. The posterior density function of θ is as follow:

$$\pi_1(\theta \mid \alpha, \mathrm{d}ata) = \Gamma(a+d, \sum_{i=1}^{d} x_{i:n}^{\alpha} + (n-d)c^{\alpha} + b).$$

Also, the conditional density of α, given data, is log-concave (see Kundu, 2007). So, we observe that the conditional density of α given the data is

$$l(\alpha \mid data) \propto \frac{\alpha^d (\Pi_{i=1}^d x_{i:n})^{\alpha-1}\pi_2(\alpha)}{(\sum_{i=1}^{d} x_{i:n}^{\alpha} + (n-d)c^{\alpha} + b)^{a+d}}. \quad (15)$$

Now we obtain the Bayes estimates and the credible intervals of α and θ, Similarly as in Geman and Geman, (1984).

Simulation

In this section, we present some simulation results to compare the performances of the different methods proposed in the previous sections. We mainly compare the performances of the MLEs and Bayes estimations of the unknown parameters, in terms of their bias, mean squared errors ($MSEs$) and their coverage percentages. It should be mentioned that all the programs are written in R.

In each case, we generated a sample from Weibull distribution with $\alpha = 1$, $\lambda = 1$ and $n = 50$. The simulation is carried out for different choices of k, r, T_1 and T_2 values. We have estimated the α and $\theta\,(\frac{1}{\lambda^{\alpha}})$ using the MLEs. For computing the MLEs, we have used the EM algorithm and computed the coverage percentages of the confidence intervals using the observed Fisher information matrix.

We have estimated the α and θ using the Bayes estimates. For computing the Bayes estimators, it is assumed that α and θ have $\Gamma(a,b)$ and $\Gamma(c,d)$ priors, respectively. Moreover, we used the non-informative gamma priors for both the shape and scale parameters, that is, when the hyper parameters are 0, ($a = b = c = d = 0$). The Bayes estimators are computed under the squared error loss function and with respect to the above priors. For comparison purposes, we also compute the 95% HPD credible intervals from the Gibbs samples. We replicate the process 10000 times and report the Bias, MSEs and coverage percentages in Tables 1-6.

From Tables 1-6 the following general observations can be made. For two methods, (i) for fixed r, k and T_2 when T_1 increases from 0.1 to 2, the bias and *MSEs* decrease (Tables 1 and 2) and the coverage percentages increase (Table 3), (ii) for fixed r, k and T_1 when T_2 increases from 0.5 to 4, the bias, *MSEs* decrease (Tables 4 and 5) and the coverage percentages increase (Table 6).

The bias and *MSEs* of the Bayes estimators are marginally larger than MLEs for small T_1 or T_2 but for large T_1 or T_2 thay are more similar. The coverage percentages of the credible intervals are usually larger than the confidence intervals in most cases, because the average credible lengths are larger than the average confidence lengths in all the cases considered.

Also, for computing bias, *MSEs* and 95% HPD credible intervals for Bayes estimators, other than non-informative priors, we also used informative priors. We have taken the following hyper parameters for informative priors $a = 0.9, b = 0.8$ and $c = 1.05, d = 1$, the results are reported for α and θ parameters in Tables 7 and 8, respectively. Comparing the two Bayes estimators based on non-informative priors ($Bayes_{no}$) and informative priors ($Bayes_w$) shows that in some schemes the Bayes estimators based on informative priors perform better than the Bayes estimators based on non-informative priors in terms of bias, *MSEs* and 95% HPD credible intervals, and in the most schemes their performances are similar.

Table 1. *Bias of the (α, θ) for Unified HCS when T_2 is 2.3*

r	k		T_1			
			0.1	0.8	1.5	2
17	5	MLE	(0.1203,0.2950)	(0.0348,0.0259)	(0.0273,0.0163)	(0.0245,0.0133)
		Bayes	(0.1838,0.4928)	(0.0669,0.0424)	(0.0434,0.0116)	(0.0389,0.0051)
24	11	MLE	(0.0743,0.0119)	(0.0356,0.0338)	(0.0269,0.0154)	(0.0220,0.0105)
		Bayes	(0.1153,0.1536)	(0.0672,0.0445)	(0.0434,0.0089)	(0.03629,0.0085)
47	11	MLE	(0.0245,0.0169)	(0.0271,0.0143)	(0.0262,0.0146)	(0.0243,0.0098)
		Bayes	(0.0383,0.0071)	(0.0403,0.0074)	(0.0373,0.0079)	(0.0375,0.0054)
	29	MLE	(0.0256,0.0135)	(0.0255,0.0134)	(0.0274,0.0126)	(0.0252,0.0142)
		Bayes	(0.0374,0.0111)	(0.0396,0.0103)	(0.0408,0.0066)	(0.0374,0.0089)
	41	MLE	(0.0237,0.0133)	(0.0242,0.0151)	(0.0274,0.0141)	(0.0253,0.0126)
		Bayes	(0.0410,0.0084)	(0.0386,0.0075)	(0.0393,0.0105)	(0.0411,0.0063)

Table 2. *MSEs of the* (α,θ) *for Unified HCS when* T_2 *is 2.3*

r	k		T_1			
			0.1	0.8	1.5	2
17	5	MLE	(0.0953,0.8715)	(0.0356,0.0461)	(0.0223,0.0282)	(0.0185,0.0259)
		Bayes	(0.1278,2.1021)	(0.0410,0.0489)	(0.0239,0.0289)	(0.0202,0.0259)
24	11	MLE	(0.0519,0.1332)	(0.0348,0.0450)	(0.0218,0.0288)	(0.0186,0.0257)
		Bayes	(0.0647,0.1771)	(0.0406,0.0458)	(0.0234,0.0287)	(0.0193,0.0025)
47	11	MLE	(0.0178,0.0269)	(0.0184,0.0259)	(0.0185,0.0264)	(0.0178,0.0262)
		Bayes	(0.0187,0.0257)	(0.0196,0.0230)	(0.0188,.0265)	(0.0183,0.0258)
	29	MLE	(0.0181,0.0264)	(0.0181,0.0259)	(0.0180,0.0263)	(0.0180,0.0257)
		Bayes	(0.0185,0.0272)	(0.0195,0.0257)	(0.0197,0.0263)	(0.0184,0.0254)
	41	MLE	(0.0178,0.0269)	(0.0176,0.0262)	(0.0179,0.0269)	(0.0176,0.0259)
		Bayes	(0.0192,0.0261)	(0.0187,0.0261)	(0.0191,0.0269)	(0.0196,0.0258)

Table 3. *Coverage Percentages of the* (α,θ) *for Unified HCS when* T_2 *is 2.3*

r	k		T_1			
			0.1	0.8	1.5	2
17	5	MLE	(0.5784,0.5207)	(0.7757,0.6862)	(0.8669,0.8865)	(0.8732,0.9126)
		Bayes	(0.9112,0.9282)	(0.9419,0.9471)	(0.9449,0.9419)	(0.9405,0.9429)
24	11	MLE	(0.7227,0.6519)	(0.7898,0.6931)	(0.8724,0.8815)	(0.8715,0.9141)
		Bayes	(0.9273,0.9347)	(0.9444,0.9536)	(0.9452,0.9431)	(0.9463,0.9474)
47	11	MLE	(0.8636,0.9038)	(0.8601,0.9066)	(0.8599,0.8996)	(0.8695,0.9053)
		Bayes	(0.9440,0.9404)	(0.9390,0.9432)	(0.9449,0.9394)	(0.9463,0.9410)
	29	MLE	(0.8622,0.9004)	(0.8647,0.9023)	(0.8676,0.9014)	(0.8625,0.9020)
		Bayes	(0.9465,0.9379)	(0.9387,0.9406)	(0.9400,0.9409)	(0.9427,0.9466)
	41	MLE	(0.8631,0.8920)	(0.8661,0.8975)	(0.8631,0.8934)	(0.8653,0.8965)
		Bayes	(0.9449,0.9411)	(0.9435,0.9393)	(0.9406,0.9402)	(0.9402,0.9413)

Table 4. *Bias of the* (α,θ) *for Unified HCS when* T_1 *is 0.2*

r	k		T_2			
			0.5	1	2.5	4
29	11	MLE	(0.0514,0.6864)	(0.0501,0.0672)	(0.0492,0.0656)	(0.0428,0.0719)
		Bayes	(0.1036,0.1317)	(0.0885,0.0865)	(0.0899,0.0896)	(0.0977,0.0913)
	21	MLE	(0.0720,0.9822)	(0.0568,0.0667)	(0.0609,0.0668)	(0.0627,0.0741)
		Bayes	(0.1200,0.1520)	(0.0887,0.0832)	(0.0929,0.0926)	(0.0976,0.0903)
38	11	MLE	(0.0489,0.0681)	(0.0299,0.0226)	(0.0443,0.0372)	(0.0431,0.0223)
		Bayes	(0.1018,0.1339)	(0.0564,0.0267)	(0.0634,0.0330)	(0.0660,0.0327)
	33	MLE	(0.0483,0.0467)	(0.0420,0.0366)	(0.0414,0.0320)	(0.0424,0.0319)
		Bayes	(0.0790,0.0565)	(0.0673,0.0374)	(0.0648,0.0321)	(0.0611,0.0312)
41	39	MLE	(0.0456,0.0320)	(0.0422,0.0289)	(0.0385,0.0264)	(0.0390,0.0263)
		Bayes	(0.0604,0.0289)	(0.0591,0.0271)	(0.0577,0.0261)	(0.0553,0.0226)

Table 5. *MSEs of the* (α,θ) *for Unified HCS when* T_1 *is 0.2*

r	k		T_2			
			0.5	1	2.5	4
29	11	MLE	(0.0583,0.0108)	(0.0404,0.0688)	(0.0393,0.0654)	(0.0400,0.0688)
		Bayes	(0.0745,0.1384)	(0.0459,0.0809)	(0.0458,0.0841)	(0.0482,0.0814)
	21	MLE	(0.0573,0.0971)	(0.0392,0.0704)	(0.0402,0.0644)	(0.0406,0.0716)
		Bayes	(0.0712,0.1288)	(0.0466,0.0814)	(0.0473,0.0838)	(0.0482,0.0797)
38	11	MLE	(0.0595,0.1086)	(0.0302,0.0381)	(0.0253,0.0345)	(0.0251,0.0340)
		Bayes	(0.0736,0.1408)	(0.0339,0.0404)	(0.0283,0.0364)	(0.0294,0.0368)
	33	MLE	(0.0319,0.0459)	(0.0284,0.0353)	(0.0248,0.0349)	(0.0249,0.0328)
		Bayes	(0.0377,0.0505)	(0.0334,0.0371)	(0.0288,0.0367)	(0.0277,0.0362)
41	39	MLE	(0.0248,0.0332)	(0.0244,0.0314)	(0.0217,0.0306)	(0.0220,0.0302)
		Bayes	(0.0270,0.0343)	(0.0269,0.0333)	(0.0245,0.0319)	(0.0248,0.0313)

Table 6. *Coverage Percentages of the* (α,θ) *for Unified HCS when* T_1 *is 0.2*

r	k		T_2			
			0.5	1	2.5	4
29	11	MLE	(0.6322,0.5739)	(0.7800,0.7307)	(0.7918,0.7486)	(0.7988,0.7441)
		Bayes	(0.9420,0.9471)	(0.9448,0.9351)	(0.9332,0.9349)	(0.9301,0.9344)
	21	MLE	(0.6854,0.6354)	(0.7769,0.7237)	(0.7917,0.7520)	(0.7935,0.7432)
		Bayes	(0.9353,0.9503)	(0.9339,0.9361)	(0.9305,0.9351)	(0.9300,0.9350)
38	11	MLE	(0.6320,0.5668)	(0.8181,0.7596)	(0.8611,0.9127)	(0.8664,0.9150)
		Bayes	(0.9418,0.9435)	(0.9409,0.9388)	(0.9384,0.9392)	(0.9380,0.9387)
	33	MLE	(0.8390,0.9626)	(0.8533,0.9716)	(0.8627,0.9154)	(0.8665,0.9185)
		Bayes	(0.9310,0.9369)	(0.9394,0.9429)	(0.9363,0.9386)	(0.9414,0.9411)
41	39	MLE	(0.8592,0.9102)	(0.8646,0.9163)	(0.8666,0.8961)	(0.8630,0.8979)
		Bayes	(0.9410,0.9402)	(0.9484,0.9402)	(0.94815,0.9398)	(0.9587,0.9415)

Table 7. *Bias* $(MSEs)$ *Coverage Percentages of the* α *for Unified HCS*

r	k		$T_1=0.2, T_2=0.7$	$T_1=0.5, T_2=1.5$	$T_1=1, T_2=1.5$	$T_1=1.2, T_2=2.5$
19	11	$Bayes_{no}$	0.105(0.0747)0.945	0.053(0.0494)0.961	0.027(0.0283)0.942	0.0188(0.0232)0.955
		$Bayes_{w}$	0.084(0.0553)0.967	0.061(0.0492)0.962	0.033(0.0288)0.945	0.017(0.0235)0.958
21	7	$Bayes_{no}$	0.080(0.0566)0.954	0.069(0.0537)0.949	0.023(0.0258)0.958	0.028(0.0255)0.948
		$Bayes_{w}$	0.069(0.0472)0.965	0.074(0.0516)0.950	0.028(0.0277)0.954	0.023(0.0227)0.955
	18	$Bayes_{no}$	0.086(0.0606)0.955	0.064(0.0481)0.960	0.023(0.0272)0.952	0.015(0.0225)0.950
		$Bayes_{w}$	0.068(0.0464)0.967	0.059(0.0486)0.954	0.029(0.0289)0.950	0.018(0.0225)0.955
38	19	$Bayes_{no}$	0.033(0.0382)0.959	0.033(0.0226)0.953	0.036(0.0242)0.951	0.043(0.0250)0.948
		$Bayes_{w}$	0.040(0.0373)0.942	0.036(0.0258)0.937	0.035(0.0237)0.952	0.036(0.0219)0.945

Table 8. *Bias* $(MSEs)$ *Coverage Percentages of the* θ *for Unified HCS*

r	k		$T_1 = 0.2, T_2 = 0.7$	$T_1 = 0.5, T_2 = 1.5$	$T_1 = 1, T_2 = 1.5$	$T_1 = 1.2, T_2 = 2.5$
19	11	$Bayes_{no}$	0.246(0.5831)0.946	0.093(0.0976)0.956	0.009(0.0341)0.944	0.025(0.0346)0.947
		$Bayes_w$	0.186(0.2045)0.947	0.094(0.0853)0.953	0.015(0.0328)0.952	0.020(0.0304)0.944
21	7	$Bayes_{no}$	0.154(0.1976)0.944	0.116(0.1061)0.953	0.012(0.0325)0.954	0.017(0.0308)0.949
		$Bayes_w$	0.118(0.1218)0.960	0.106(0.0879)0.954	0.0175(0.0343)0.941	0.016(0.0283)0.956
	18	$Bayes_{no}$	0.181(0.2701)0.936	0.102(0.0937)0.959	0.019(0.0369)0.939	0.011(0.0297)0.948
		$Bayes_w$	0.124(0.1272)0.954	0.089(0.0811)0.954	0.025(0.0337)0.956	0.017(0.0298)0.944
38	19	$Bayes_{no}$	0.032(0.0522)0.962	0.026(0.0354)0.941	0.032(0.0344)0.940	0.028(0.0313)0.951
		$Bayes_w$	0.061(0.0599)0.947	0.031(0.0343)0.944	0.021(0.0313)0.948	0.032(0.0267)0.951

Data Analysis

In this section, we present one example to illustrate the methods of inference developed in the preceding section. The data set is from Lawless (1982, page 228) that were given by Thoman et al. (1969), who attributed them to test on the endurance of deep-groove ball bearings discussed by Lieblein and Zelen (1956). The observations are the number of million revolutions before failure for each of 23 balls bearings, the individual bearings were inspected periodically to determine whether "failure" had occurred, but we treat the failure times as continuous. The 23 failure times are: 17.88, 28.92, 33, 41.52, 42.12, 45.6, 48.4, 51.84, 51.96, 54.12, 55.56, 67.9, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 127.04, 173.4.

One question arises whether the data fit Weibull distribution or not. To check for goodness-of-fit we compute the Anderson-Darling statistic, it is 0.329 and the associated p value is 0.250. Since the p value is quite high, we cannot reject the null hypothesis that the data are coming from the Weibull distribution.

We have created six artificially unified HCS data sets from the above uncensored data set, and consider the following sampling schemes 1-6.

1: $T_1 = 90, T_2 = 100, k = 14, r = 16.$ 2: $T_1 = 90, T_2 = 105, k = 14, r = 17.$

3: $T_1 = 70, T_2 = 95, k = 14, r = 18.$ 4: $T_1 = 60, T_2 = 95, k = 12, r = 15.$

5: $T_1 = 60, T_2 = 100, k = 14, r = 19.$ 6: $T_1 = 70, T_2 = 85, k = 17, r = 21.$

For schemes 1-6 we have estimated the unknown parameters using the MLEs and the Bayes ($Bayes_{no}$ and $Bayes_w$) estimations. The estimations for α and θ are reported in Tables 9 and 10, respectively.

Based on the uncensored sample the MLEs of α and $\theta = 1/\lambda^{\alpha}$ are 2.103 and 9.5×10^{-5}, respectively. For computing the Bayes estimators, we mainly consider squared error loss functions and gamma priors on both α and θ same as the previous section. Based on the above assumptions we obtain the non-informative Bayes estimators of α and θ as 2.143 and 2.1×10^{-4}, respectively. also, the informative Bayes estimators of α and θ as 2.08 and 2.9×10^{-4}, respectively.

From Tables 9 and 10, it is observed that the values of T_1 and T_2 play a major role for the estimations of α and θ, the scheme 4, shows this fact.

Table 9. *The MLEs and Bayes Estimations of the α for Schemes 1-6*

Schemes	*MLEs*	*Bayes Estimations*	
		$Bayes_{no}$	$Bayes_{w}$
1	2.253	2.348	2.214
2	2.293	2.365	2.272
3	2.240	2.318	2.222
4	3.188	3.384	3.003
5	2.240	2.304	2.210
6	2.293	2.383	2.254

Table 10. *The MLEs and Bayes Estimations, of the θ for Schemes 1-6*

Schemes	*MLEs*	*Bayes Estimations*	
		$Bayes_{no}$	$Bayes_{w}$
1	0.000051	0.00021	0.00028
2	0.000044	0.00021	0.00022
3	0.000054	0.00019	0.00025
4	0.000001	0.00002	0.00005
5	0.000054	0.00019	0.00026
6	0.000044	0.00017	0.00024

Conclusions

In this paper, we have considered the classical and Bayesian inference procedures for Weibull parameters based on the unified HCS. It is observed that the maximum likelihood estimates can be obtained by solving two non-linear equations, but they can not be obtained in closed form. We proposed to use the EM algorithm to compute the MLEs. We also obtain the Bayes estimates of the unknown parameters under the assumption of independence using the Gibbs sampling procedure. We compared The performances of the Bayes estimators under the assumption of the non-informative priors with the corresponding MLEs and found that their behaviors were similar, as expected. Also, we compared the performances of the Bayes estimates based on informative priors and non-informative priors. Finally, as an illustration, we have presented one numerical example to carry out the performance of the procedures obtained.

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