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Stochastic Forecasting of Demographic Components Based on Principal Component Analyses

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Stochastic Forecasting of Demographic Components Based on Principal Component Analyses

Patrizio Vanella

Abstract

Adequate forecasts of future population developments that are based on cohort-component methods demand an age- and sex-specific analysis; otherwise, the structure of the future population cannot be specified correctly. Age-specific demographic measures are both highly correlated and highly dimensional. Thus, a methodology that not only considers the correlations between the random variables but also reduces the effective dimensionality of the forecasting problem is needed: principal component analysis serves both purposes simultaneously. This study presents principal component analysis, from a mathematical-statistical perspective, to users from the field of population studies. Furthermore, important aspects of time series analysis, which are vital for an accurate stochastic forecast, are explained. The application is illustrated via the simultaneous projection of selected age- and sex-specific survival rates with projection intervals for Germany, Italy, Austria, and Switzerland.

Keywords: Forecasting, Multivariate Methods, Principal Component Analysis, Quantitative Population Studies, Time Series Analysis.

Introduction and Motivation

Official population projections are commonly conducted on the basis of deterministic cohort-component models (Alho, 1990; Pötzsch and Rößger, 2015). Stochastic forecasts are favorable compared to deterministic approaches (Keilman and Pham, 2000) since, in addition to the most probable scenario, they identify and quantify with respective probabilities infinitely many possible scenarios. Stochastic models may also be based on the components of fertility, migration and mortality. Autocorrelation and cross-correlation must be considered in future forecasts because there are correlations between the different age groups and genders as well as between observations at different points in the time series. Therefore, this paper first presents a short introduction to principal component analysis (PCA), where the focus is on explaining its use and a short illustration of its functionality.

Moreover, important concepts of time series analysis (TSA) are considered, with the explanation restricted to the aspects that are required for practical applications, with no claim on completeness in mind. This contribution therefore may be understood as a guide for statistical offices or demographic research institutes; the concrete projections serve only illustrative purposes and should not be mistaken as actual forecasts for future development. The methods are not restricted to population studies; forecasters from various disciplines might find this contribution interesting reading material. The explanation of the method is the focal point of the paper, especially the implementation of the illustrated statistical concepts of modeling and forecasting the components of demographic developments. The method is demonstrated by simulating selected age- and sex-specific survival rates (ASSSRs) for Germany, Italy, Austria, and Switzerland, but the model could be applied to other countries or regions as well. Application to other problems in demography or in other fields is not shown but may be done in principle the same way it is presented in this contribution.

Introduction to Principal Component Analysis

Detailed population forecasts by sex and age show a high degree of dimensionality, since for two genders up to 116 age groups should be investigated (see Vanella, 2017). Moreover, these quantities are highly correlated with each other. Forecasters have to address these two problems with appropriate methods. PCA is recommended because it simultaneously addresses both problems. PCA was originally developed by Pearson (1901) geometrically and involves applying an orthogonal transformation to the original variables into the same number of new, uncorrelated variables, which are labeled principal components (PCs). The method is especially well suited for situations in which no causal relationship between the variables is quantified, as is the case for regression analyses. Therefore, PCA is especially appropriate for forecasting age- and sex-specific measures in a demographic context. Each PC is a linear combination of N original variables. Let

$S_{i,t}$ be the ith ASSSR in period t. Then, the jth PC $P_{j,t}$ in the same period is calculated by the following (Chatfield and Collins, 1980):

$$P_{j,t} = \sum_{i=1}^{N} e_{i,j,t} S_{i,t} =: \overrightarrow{e_{j,t}^{T}} \overrightarrow{S_t}, \tag{1}$$

where $e_{i,j,t}$ can be interpreted as a correlation coefficient between the ith ASSSR and the jth PC in period t.

Within the PCA framework, the PCs are deduced in decreasing order of the magnitude of the total variance they explain. This means the first PC explains the largest share of the variance in the original variables. Through the transformation, a complex system with many variables can effectively be reduced to few dimensions since the first few PCs explain the majority of the variance.

The first principal component (PC 1) is chosen to explain as much of the variance as possible. Statistically, this means that the coefficients (or *loadings*, as they are also called in this context) of the first linear combination are adjusted to maximize the amount of covariance in the original variables that is explained. The calculation is now illustrated with ASSSRs. For simplicity, the loadings are assumed to be invariant through time;[1] thus the index t is omitted. Given the covariance matrix of $\boldsymbol{S}$ (the matrix of all ASSSRs), labeled $\boldsymbol{\Sigma}$, the variance of PC 1 is given by equation (2):

$$Var[P_1] = Var\left[\overrightarrow{e_1^T}\boldsymbol{S}\right] = \overrightarrow{e_1^T}\boldsymbol{\Sigma}\overrightarrow{e_1}. \tag{2}$$

The vector $\overrightarrow{e_1}$ can be chosen arbitrarily. To reach a unique solution of the maximization problem, a restriction for the elements of $\overrightarrow{e_1}$ (also called the *eigenvector*) must be stated. Normalizing $\overrightarrow{e_1}$ to a length of one ensures an orthogonal transformation. A vector has length one if its scalar product with itself is one (Handl, 2010):

$$\overrightarrow{e_1^T}\overrightarrow{e_1} = 1. \tag{3}$$

Due to the method of Lagrange multipliers, the stationary points[2] of a function $f(\vec{x})$ under condition $g(\vec{x}) = c$ can be identified through the identification of the stationary points of the affiliated Lagrange function $\mathcal{L}(\vec{x},\lambda)$. The Lagrangiana is defined as follows:

$$\mathcal{L}(\vec{x},\lambda) = f(\vec{x}) - \lambda[g(\vec{x}) - c] \tag{4}$$

Therefore, the maximization problem for the variance can be solved by finding the stationary point of the following Lagrange function:

---

[1]Hyndman and Ullah (2007) have proposed a different approach with variable loadings.
[2]Those might be either local minima, maxima or saddle points.

$$\mathcal{L}(\vec{x},\lambda) = \overrightarrow{e_1^T}\boldsymbol{\Sigma}\overrightarrow{e_1} - \lambda[\overrightarrow{e_1^T}\overrightarrow{e_1} - 1]. \tag{5}$$

Accordingly, the stationary point is determined as follows:

$$\frac{\partial\mathcal{L}(\overrightarrow{e_1},\lambda)}{\partial\overrightarrow{e_1}} = 2\boldsymbol{\Sigma}\overrightarrow{e_1} - 2\lambda\overrightarrow{e_1} = 2(\boldsymbol{\Sigma} - \lambda\boldsymbol{I})\overrightarrow{e_1} = \vec{0} \wedge$$
$$1 - \overrightarrow{e_1^T}\overrightarrow{e_1} = 0. \tag{6}$$

Here, $\boldsymbol{I}$ is an identity matrix, which for $\overrightarrow{e_1}$, consisting of p elements with dimensions $p \times p$, becomes the following:

$$\boldsymbol{I} := \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}. \tag{7}$$

The first equation in (6) indicates that the matrix on the left-hand side of the equation must be singular. Since $\overrightarrow{e_1}$ must not be a null vector, guaranteeing a nontrivial solution, it follows that the determinant of the matrix $\boldsymbol{\Sigma} - \lambda\boldsymbol{I}$ has to equal zero (Chatfield and Collins, 1980):

$$|\boldsymbol{\Sigma} - \lambda\boldsymbol{I}| = 0. \tag{8}$$

This is illustrated with a practical example. From age- and sex-specific data on deaths and the end-of-year population, provided by the federal statistical offices of Germany, Italy, and Austria and complemented by downloads from the Human Mortality Database and the Eurostat Database, the ASSSRs for Germany, Italy, Austria, and Switzerland are calculated for the years 1952-2016 (Destatis, 2005, 2015a, 2015b, 2015c, 2016, 2017a, 2017b, 2018a, 2018b; Eurostat, 2018; Human Mortality Database, 2018a, 2018b, 2018c, 2018d; Istat, 2018a, 2018b, 2018c, 2018d, 2018e, 2018f; STATcube, 2018). For illustration, PCA is performed on the covariance matrix of the logit transformed ASSSRs of 25-year-old (cohort based) males for the four mentioned countries for the years 1952-2016. Survival rates can only take values greater than zero and less than one, so their projections are made through simulation of their logits. A logistic transformation of an ASSSR *s* can be calculated as follows (Johnson, 1949):

$$logit(s) = \ln\left(\frac{s}{1-s}\right) \tag{9}$$

The transformation leads to new unrestricted variables, whereas the underlying ASSSRs cannot take simulation values outside the open interval (0,1) in the forecast. After simulation, the results must be transformed back through the inverse logit to obtain the final ASSSR trajectories.

The solutions of the optimization problem in this case are approximately $\lambda_1 \approx 0.617, \lambda_2 \approx 0.031, \lambda_3 \approx 0.013$, and $\lambda_4 \approx 0.007$, which are also called the eigenvalues (*EWs*) of the covariance matrix. The sum of the EWs is equal to the sum of the covariance of the original variables; therefore, the EWs are sorted in decreasing order. As mentioned earlier, one of the two reasons to apply PCA is to reduce the original statistical problem into a small number of variables that explain as much of the covariance in the original variables as possible. This means the first eigenvalue (EW) represents the variance that is explained by PC 1. Therefore, we can derive that PC 1 explains approximately 92.4% of the overall covariance of the four time series (TS), whereas PC 1 and PC 2 already explain over 97%.

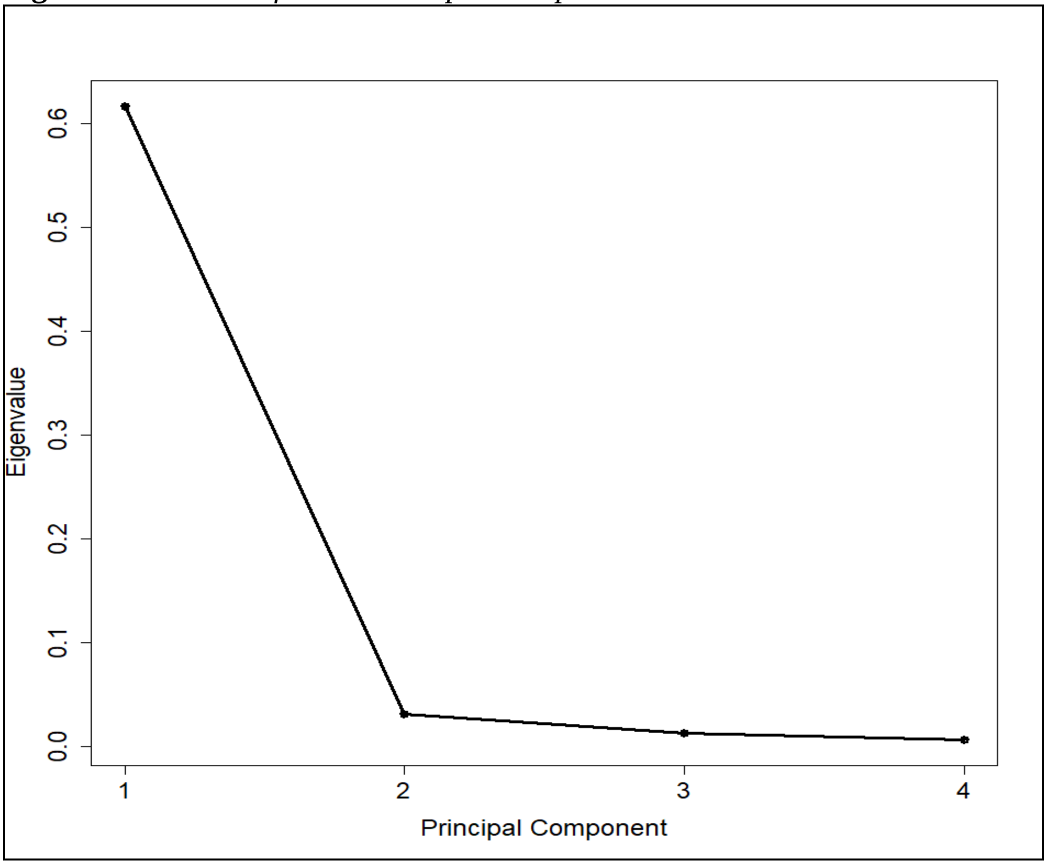
Plugging the EWs into (6) individually leads to the respective EVs, e.g., EV 1:

$$\overrightarrow{e_1} \approx \begin{bmatrix} -0.522 \\ -0.414 \\ -0.539 \\ -0.516 \end{bmatrix}.$$

PC 1 is negatively loaded with all of the four logit-ASSSRs, therefore indicating a type of general mortality index, similar to the Lee-Carter index (Lee and Carter, 1992). The associated PCs can be easily derived by (1).

One important question is the determination of the number of PCs for the analysis. There is no trivial answer; the determination of the number of PCs to use is subjective. Nevertheless, criteria have been proposed to simplify the decision. One possibility is to define a minimum percentage of the variation to be explained. If we would, e.g., target covering at least 95% of the variance in the ASSSRs, we would take the first two PCs into our model. Another common method to select the number of PCs is graphically analyzing the EVs of the covariance matrix with a scree plot (see Handl, 2010), as shown in Figure 1.

Figure 1. *Scree Plot for the Principal Components*



Only the PCs that lie on the left-hand side of the elbow are included in the model; moreover, there is no clear consensus whether the PC at the elbow itself should be included as well. In this case, the scree plot suggests one or two PCs. From a practitioner's point of view, it is generally worthwhile to include the PC at the elbow when making forecasts; otherwise, in many cases, a relatively large share of the variance would be ignored, leading to biased results when constructing prediction intervals (PIs). This result can be observed in many practical applications. Vanella (2017) proposed a simulation method that includes the uncertainty arising from omitting most of the PCs to prevent excessively narrow forecast PIs. This topic is not considered further in this paper.[3] Kaiser's and Jolliffe's criteria are additional alternatives. Kaiser's criterion suggests using only PCs with EWs that are larger than the mean EW (Handl, 2010). Jolliffe (2002) proposed 70% of the mean as the lower limit. Nevertheless, the choice of criterion is subjective.

The focus of this section was the general description of PCA in a semimanual practical application for a better understanding of the method. Nevertheless, PCA can be performed relatively easily using **R**[4].

[3]For further reading on this issue, see the aforementioned article.
[4]The standard commands **prcomp** and **princomp**, which are pre-installed, can be used for this.

Main Features of Time Series Analysis

In the section, some aspects of TSA, which are highly relevant in the context of PC forecasting, will be explained.

A TS is a variable that generates one observation in each period. The fundamental concept of modern TSA is stationarity, which will be explained briefly. The TS of the ASSSR (in period t) of the 25-year-old males in Germany is defined as a_t. Stationarity (also called weak stationarity in the literature) is sufficiently defined by two conditions: mean stationarity and auto covariance stationarity (Shumway and Stoffer, 2011). The mean stationarity is defined by the equality of the TS mean in each period:

$$E[a_s] = E[a_t] \; \forall \; s, t. \tag{10}$$

Autocovariance stationarity means the theoretical auto covariance between two observations of the TS does not depend on the point in time but on the length of the time interval separating the two observations:

$$Cov[a_{t+h}, a_t] = Cov[a_h, a_0] \forall \; t. \tag{11}$$

Autoregressive integrated moving average (ARIMA) models, developed by Box and Jenkins (Box et al., 2016), are of major importance for practical applications, as subsequently explained (Shumway and Stoffer, 2011). A moving average of order q (MA(q)) is defined as

$$a_t = \omega_t - \sum_{i=1}^{q} \theta_i \omega_{t-i}, \tag{12}$$

where ω_t is a stochastic nuisance parameter in period t, which in practical applications, is normally assumed to follow a Gaussian[5] distribution with a mean of zero and a variance σ^2:

$$\omega_t \sim \mathcal{NID}(0, \sigma^2). \tag{13}$$

The stationarity assumption is beneficial because stationarity allows the assumption that the nuisance parameter is identically distributed in each period. This assumption is especially helpful for running simulations. An MA(q) model thus starts from the premise that the current observation of the variables emerges exclusively as a weighted sum of the last q manifestations of the nuisance parameter and the error in the current period. In this notation, θ_i is the correlation coefficient of the TS with respect to the error in period $t - i$. θ_i is restricted between -1 and 1:

$$|\theta_i| < 1. \tag{14}$$

[5]An alternative is to assume a Student's t-distributed nuisance parameter, as proposed by Raftery et al. (2014).

A feasible alternate representation for an MA(q) process is the *lag notation*, where L is the so-called *lag operator*[6]. The lag notation for an MA(q) process is as follows:

$$a_t = \left(1 - \sum_{i=1}^{q} \theta_i L^i\right) \omega_t. \tag{15}$$

The exponent of L indicates which past period is being considered. For example, $L^q \cdot \omega_t$ signifies ω_{t-q}.[7]

Another common type of TS model is an autoregressive model of order p (AR(p)):

$$a_t = \omega_t + \sum_{j=1}^{p} \phi_j a_{t-j}, \tag{16}$$

or in lag notation:

$$\left(1 - \sum_{j=1}^{p} \phi_j L^j\right) a_t = \omega_t. \tag{17}$$

In an AR(p) model, the TS in period t is regressed on its previous p observations (taking the error in period t into account). In this case, $L^p \cdot a_t = a_{t-p}$. Similar to the MA(q) model,

$$|\phi_j| < 1. \tag{18}$$

AR and MA models can also be combined; the combination of an AR(p) model and an MA(q) model produces an ARMA(p,q) model, which is formally defined as follows:

$$a_t = \omega_t - \sum_{i=1}^{q} \theta_i \omega_{t-i} + \sum_{j=1}^{p} \phi_j a_{t-j} \tag{19}$$

or

$$\left(1 - \sum_{j=1}^{p} \phi_j L^j\right) a_t = \left(1 - \sum_{i=1}^{q} \theta_i L^i\right) \omega_t. \tag{20}$$

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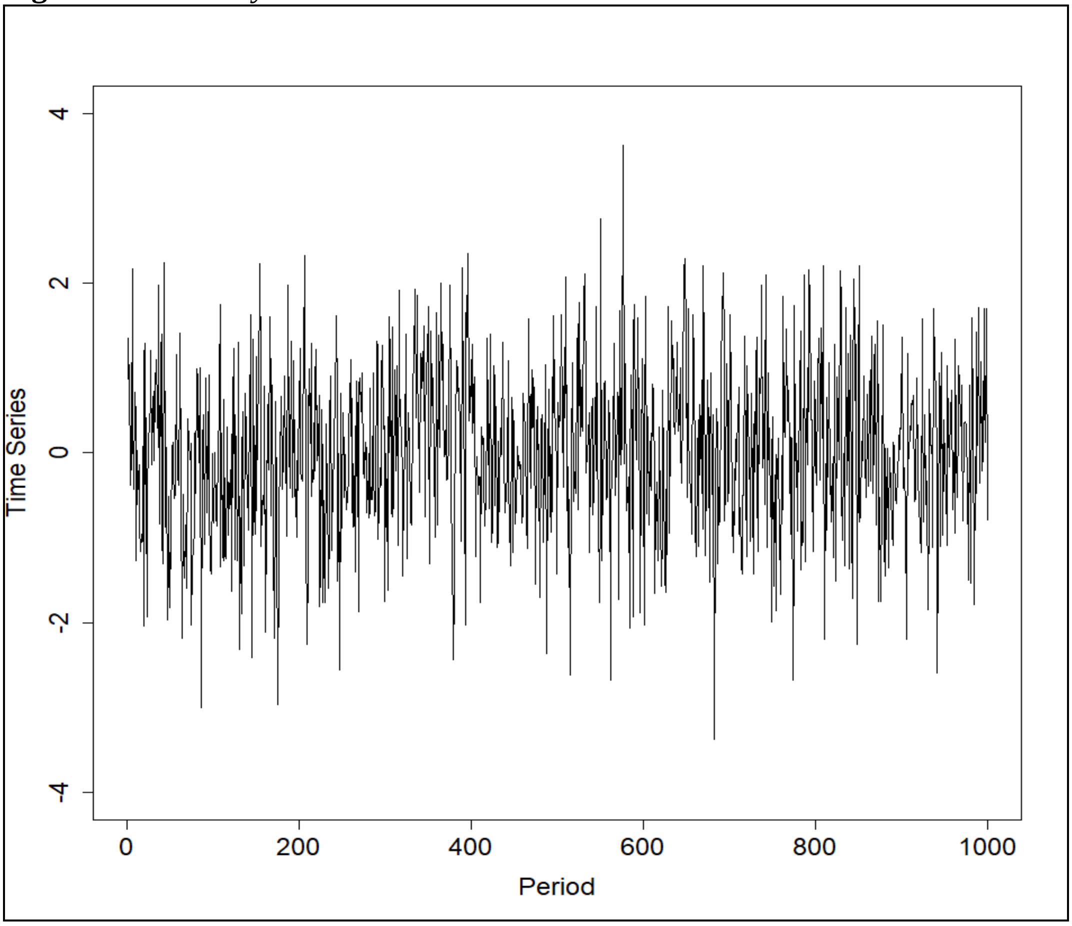
[6]Alternatively, some authors write about the *backshift operator*, which is the same.

[7]In practical applications, one has to be careful about the explicit definition of the coefficients. Some statistical packages give a slightly different output, e.g., the output in ***R*** changes the sign of the coefficient relative to this contribution.

As mentioned previously, the stationarity assumption is fundamental for ARMA processes. The question is how to identify whether a TS is stationary; graphical analysis is recommended as the first step in the investigation. Figure 2 illustrates a simulated stationary TS[8].

Figure 2. *Stationary Time Series*



It is clear that neither the mean nor the variance show trending behavior. Furthermore, the stationarity hypothesis should be confirmed using statistical tests, such as the augmented Dickey-Fuller (ADF) test and the Kwiatkowski-Phillips-Schmidt-Shin (KPSS) test. The standard ADF test checks the null hypothesis, i.e., whether for the following equation

$$a_t = \rho \cdot a_{t-1} \quad (21)$$

the condition

$H_0: \rho = 1$

[8]The TS is generated by 1,000 computer simulations of a Gaussian random variable.

holds. This condition corresponds to a *random walk* process (Dickey and Fuller, 1979). Several variants of the test exist. One variant of interest is the one with the alternative hypothesis

$$H_1: |\rho| < 1,$$

which corresponds to a stationary or asymptotically stationary process. The test statistic in this case is

$$\tau = \frac{\hat{\rho} - 1}{se(\hat{\rho})}, \tag{22}$$

matching the common Student's t-test. However, the test statistic is not compared to the quantiles of a t-distribution but to an empirical distribution produced by Dickey and Fuller from Monte Carlo simulations (Fuller, 1996).[9] In the example, τ is approximately -10.31, which means that the null hypothesis is rejected at all major significance levels. The statistical evidence indicates stationarity of the TS.

By contrast, the KPSS test is a Lagrange multiplier test with a test statistic

$$LM = \frac{\sum_{t=1}^{T} S_t^2}{SSR/T}, \tag{23}$$

where SSR is the sum of squared residuals of the regression, T is the number of periods and S_t is the sum of the residuals from the regression

$$a_t = \alpha + \beta t$$

until time t. The critical values for the underlying distribution were estimated by Kwiatkowski et al. (1992) through a Wiener process[10]. The KPSS test[11] checks the null hypothesis of stationarity for the TS. Large values lead to rejection of H_0. In the example, the test statistic is approximately 0.0625 for $H_0: \alpha = 0$ and approximately 0.0552 for $H_0: \beta = 0$, well below the critical values at all common confidence levels,[12] so the null hypothesis cannot be rejected in either case. Therefore, the KPSS test does not provide evidence against the assumption of stationarity for the random variable.

Another important test that should be considered is the ARCH-LM test for conditional heteroscedasticity. Given that our TS has the standard deviation σ_t in time *t*, the test for the equation

[9]The ADF test is implemented in common statistics software, e.g., in ***R***, using the command **adf.test** from the package **tseries** (see Trapletti and Hornik, 2018).
[10]The process grows each period by a stochastic value, which is drawn from a Gaussian random variable.
[11]The KPSS test is usually implemented in standard statistics software as well, e.g., in ***R***, using **kpss.test** from the package **tseries**.
[12]For α=0.1, the critical value is approximately 0.347 for the mean stationarity hypothesis and 0.119 for the variance stationarity hypothesis.

$$\sigma_t^2 = \alpha_0^2 + \sum_{i=1}^{p} \alpha_i^2 \epsilon_{t-i}^2$$

checks the null hypothesis

$$H_0: \alpha_i = 0 \ \forall \ i \in \mathbb{N}^+$$

where α_0^2 is some constant. If H_0 cannot be rejected, we find no evidence for heteroscedasticity in the TS[13] (Engle, 1982).

If the modeler concludes nonstationarity in the TS based on the statistical tests, a transformation is needed. In this case, it is commonly assumed that the TS was integrated, which is represented by the middle part of the ARIMA notation. A d-times-integrated TS in the simplest case is denoted as an ARIMA(0,d,0) process (Shumway and Stoffer, 2011):

$$(1 - L)^d a_t = \omega_t. \tag{24}$$

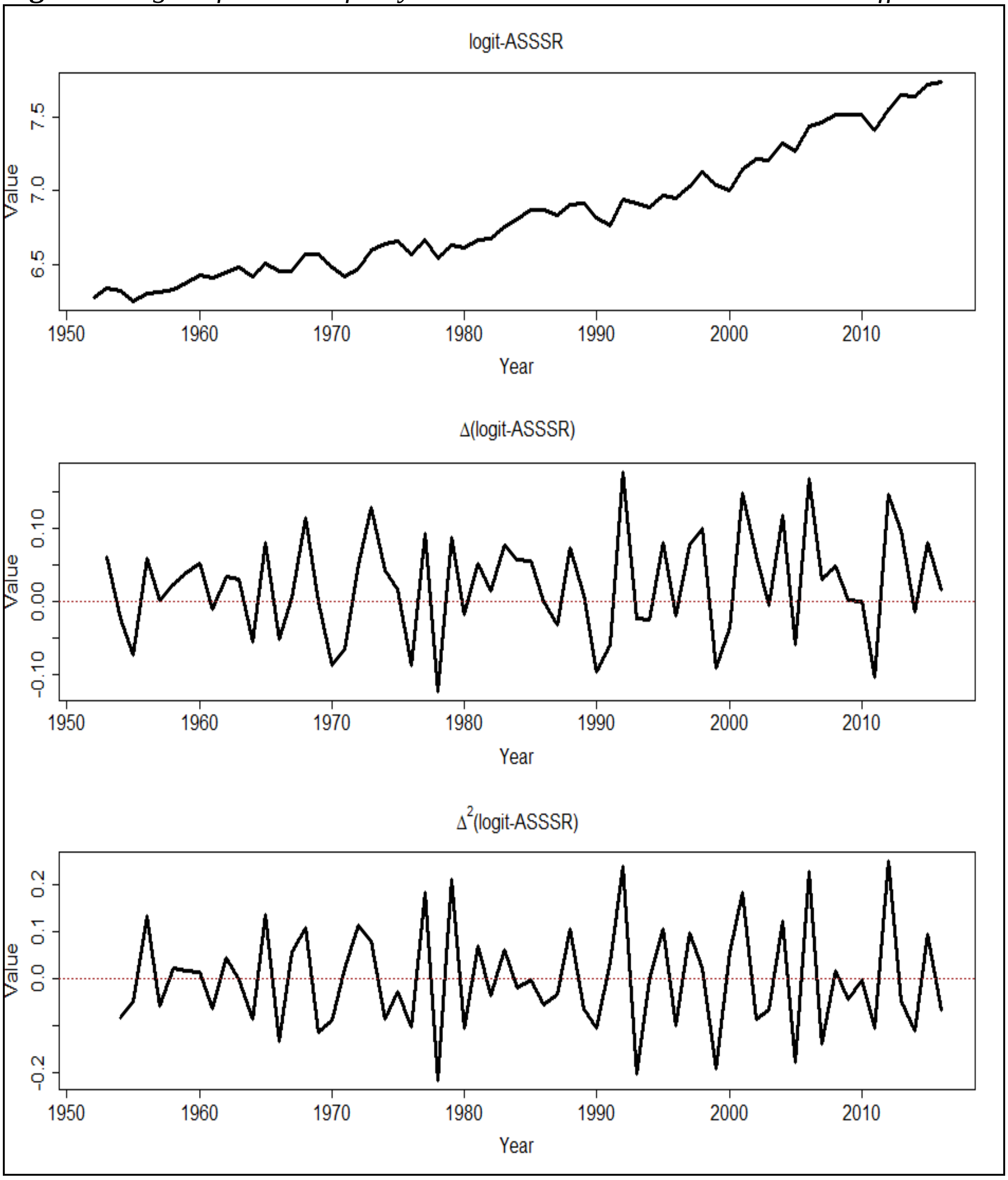
In principle, a nonstationary TS can be transformed into a stationary TS by differentiating it one or more times (Shumway und Stoffer, 2011). The first difference of a TS is calculated as follows:

$$\Delta a_t = a_t - a_{t-1}. \tag{25}$$

As known from calculus, this operation asymptotically leads to a reduction of the power of the target function (here: the TS) by one. Figure 3 illustrates the result of the differentiation by visualizing the TS of the logit-ASSSR of 25-year-old males in Germany with its first and second difference for the time horizon 1952-2016.

[13]The ARCH-LM Test should be implemented in common statistics software as well, i.e., it can be applied easily by **ArchTest**, included in the package **FinTS** (Graves, 2013).

Figure 3. *Logits of ASSSRs of 25-year-old Males with First and Second Differences*

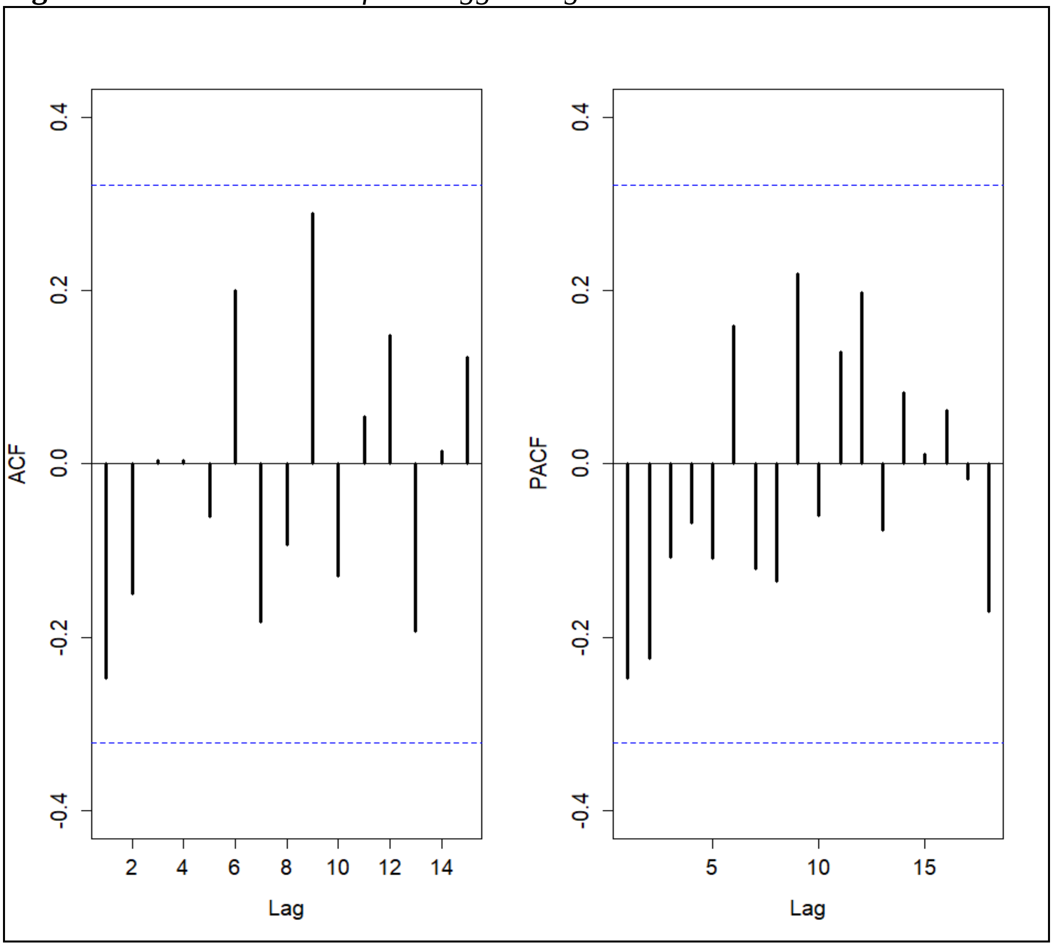


Sources: Destatis, 2005, 2015a, 2015b, 2015c, 2016, 2017a, 2017b, 2018a, 2018b; Own calculations and design.

The trending behavior of the original TS is weakened by differentiation. Graphically, we can conclude that the first difference may already be stationary, and the second difference is even smoother.

The next question to consider for a TS that has been transformed to a supposedly stationary TS is what type of ARMA model best fits the asymptotically stationary TS. Several information criteria can be applied, including Akaike's information criterion (AIC), the Bayesian information criterion (BIC), which is also known as the Schwartz criterion (see, e.g., Greene, 2012), and the Hannan-Quinn criterion (HQC) (Hannan and Quinn, 1979). These criteria follow a similar principle, relying on the loglikelihood as the goodness-of-fit measure. The difference between the criteria is the magnitude of the penalty for model complexity. The best model is the one that minimizes the criterion of choice. The specifics of the criteria are not presented here because they rely heavily on asymptotics. Thus, the reliability of the information criteria strongly depends on the length of the TS (i.e., how much data is available as base data) used as the model input. The availability and quality of data used in typical population studies, especially regarding population forecasts, are relatively poor. Therefore, the information criteria should be considered carefully. Graphical analyses based on the autocorrelation function (ACF) and the partial autocorrelation function (PACF) are recommended for investigating demographic TS[14]. For the logit-ASSSR example, both the ADF and KPSS test suggest one-time differentiation as suitable. From a practical perspective, the KPSS and ADF tests generally show poor performance for short histories, tending to mark stationarity too early. The ARCH-LM test gives a p-value of 0.5128 for the first differences-TS. Thus, the first differences are stationary with a high probability. Figure 4 shows the ACF and the PACF of the first differences for the ASSSR of 25-year-old males in Germany for the period under study.

Figure 4. *ACF and PACF of the Lagged Logit-ASSSR*



[14]For a more detailed description of the ACF and PACF, see, e.g., Shumway and Stoffer, 2011.

The graphical representations provide evidence of which lag length to choose and therefore which values of p and q are best. The graphical analysis is not trivial and requires the user to have some experience. However, some basic attributes, which ideally are observable in the figures, are associated with AR and MA processes. First, the dashed line[15] indicates the statistical significance of the lags. An AR(1) process is relatively easy to identify since the related ACF decreases exponentially, whereas the PACF has a large value for the first lag and then abruptly falls to approximately zero. An MA(1) process behaves inversely, with an exponentially decreasing PACF and an ACF with significant values for the first lag only. Figure 4 does not suggest any autocorrelation, as the estimates of the values are not statistically significant at the chosen level, which would suggest that the ASSSR-TS is simply a random walk process. This conclusion can additionally be confirmed by the information criteria.[16]

The use of the lag notation will now be explained. The general ARIMA(p,d,q) process can be described as follows:

$$\left(1-\sum_{i=1}^{p}\phi_j L^j\right)(1-L)^d a_t = \left(1-\sum_{i=1}^{q}\theta_i L^i\right)\omega_t. \tag{26}$$

In the case of an ARIMA(1,1,1) process, the lag notation is

$$(1-\phi L)(1-L)a_t = (1-\theta L)\omega_t,$$

which may be multiplied out to

$$[1-(1+\phi)L+\phi L^2]a_t = (1-\theta L)\omega_t.$$

From the definition of the lag operator, it follows that

$$a_t-(1+\phi)a_{t-1}+\phi a_{t-2} = \omega_t-\theta\omega_{t-1},$$

or equivalently,

$$a_t = (1+\phi)a_{t-1}-\phi a_{t-2}+\omega_t-\theta\omega_{t-1}.$$

Even complicated functional forms can be written in a simple way with the lag operator, which is especially helpful in the context of simulation studies in forecasting.

Forecasting Demographic Rates

This section will explain how the TS methods can be used to forecast the previously identified PCs. A first comparable approach was proposed by Bell and Monsell (1991) for forecasting age group-specific mortality in the United States. That

[15]The figures were generated in ***R*** using **acf()** and **pacf()**. The dashed lines are plotted by default, according to the chosen significance level.
[16]Standard optimization algorithms exist. In ***R***, this may be checked using **auto.arima()** in the package **forecast** (see Hyndman et al., 2018).

contribution was built upon earlier proposals by Ledermann and Breas (1959) as well as Le Bras and Tapinos (1979) for modeling and projecting age- and sex-specific mortality in France. Lee and Carter proposed a simplified version of the model of Bell and Monsell for mortality (Carter and Lee, 1992; Lee and Carter, 1992) and later fertility (Lee, 1993) forecasting. The Lee-Carter models are currently very popular in mortality and fertility forecasting. Some scientists in Germany have recently used similar models to forecast age- and sex-specific fertility and mortality rates (see, e.g., Fuchs et al., 2018; Härdle und Myšičkova, 2009; Lipps and Betz, 2005, Vanella, 2017). Deschermeier (2015) applied the Hyndman-Ullah (2007) model, which is a PCTS model adjusted for robustness and with functional PCs, allowing for the loadings explained in Section "Introduction to Principal Component Analysis" to vary over time. Vanella and Deschermeier (2018) applied a PC model for migration forecasting in Germany.

Returning to the four TS introduced in Section "Introduction to Principal Component Analysis" (ASSSRs of 25-year-old males in Germany, Italy, Austria, and Switzerland), a simple Lee-Carter model is used to estimate their future course until 2080.[17] The first step in such a forecast should be the identification of the long-term trending behavior. An accurate interpretation of the PCs for this is of high importance, since the forecaster needs to put some qualitative judgment into the initial forecast model as well. Forecasts certainly are best, when they are derived quantitatively, but can be explained qualitatively as well. In our example, the PC in Section "Introduction to Principal Component Analysis" has been identified as a general mortality index. A first graphical analysis of Figure 5 gives an idea of the long-term behavior of the PC.

[17]Disclaimer: The simulations presented here are purely of illustrative nature to show the practical application of the methods presented in Sections "Introduction to Principal Component Analysis" and "Main Features of Time Series Analysis" and should by no means be mistaken as actual forecasts.

Figure 5. *Historical Course of Mortality Index*

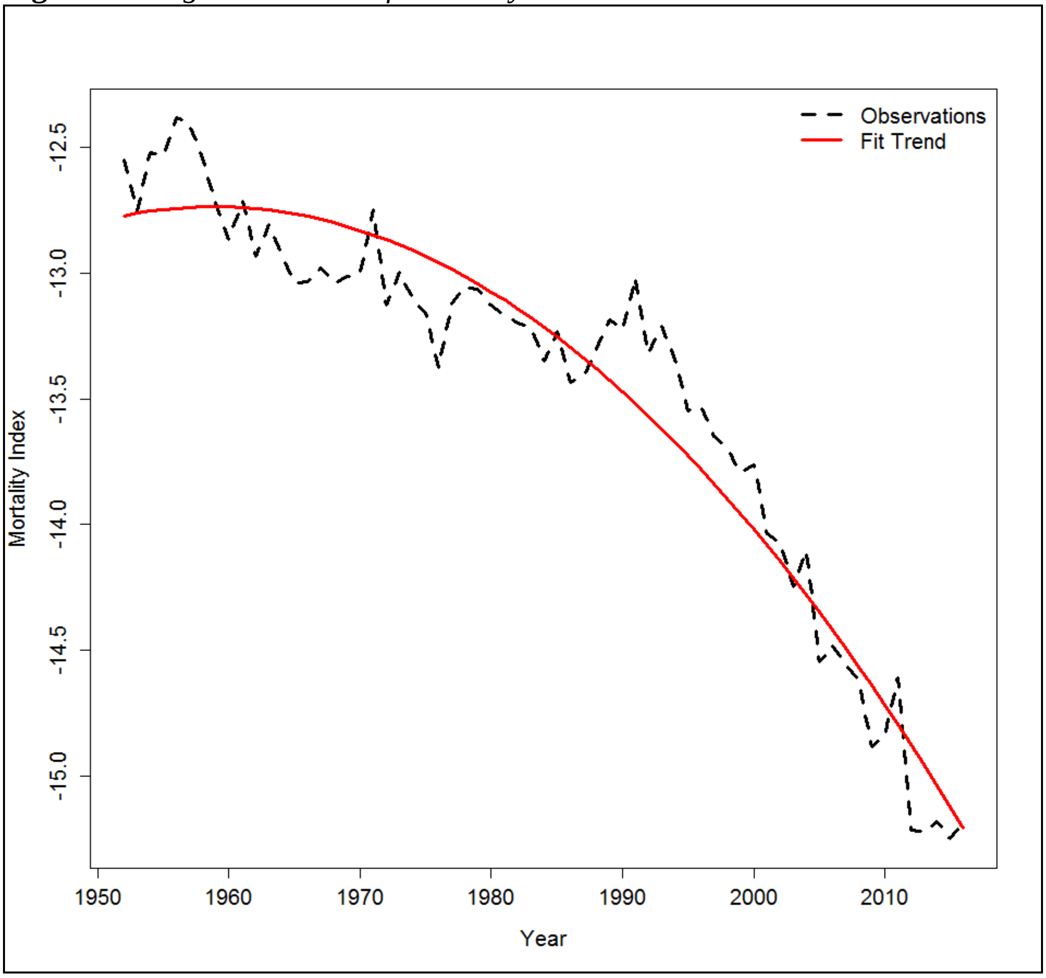
From the historical course, we might conclude a progressively decreasing course, corresponding to a clear positive trend in the survival rates. The next step is then the smoothing of the index by fitting an appropriate model by ordinary least squares (OLS) estimation[18] to the data. The fit of a quadratic model[19] renders the forecast model for the long-term trend

$$f(t) = -2916.011 + 2.964t - 0.00076t^2, \tag{27}$$

which is statistically highly significant at the individual level (for the coefficients) and due to the overall model significance. The fit is shown in Figure 6.

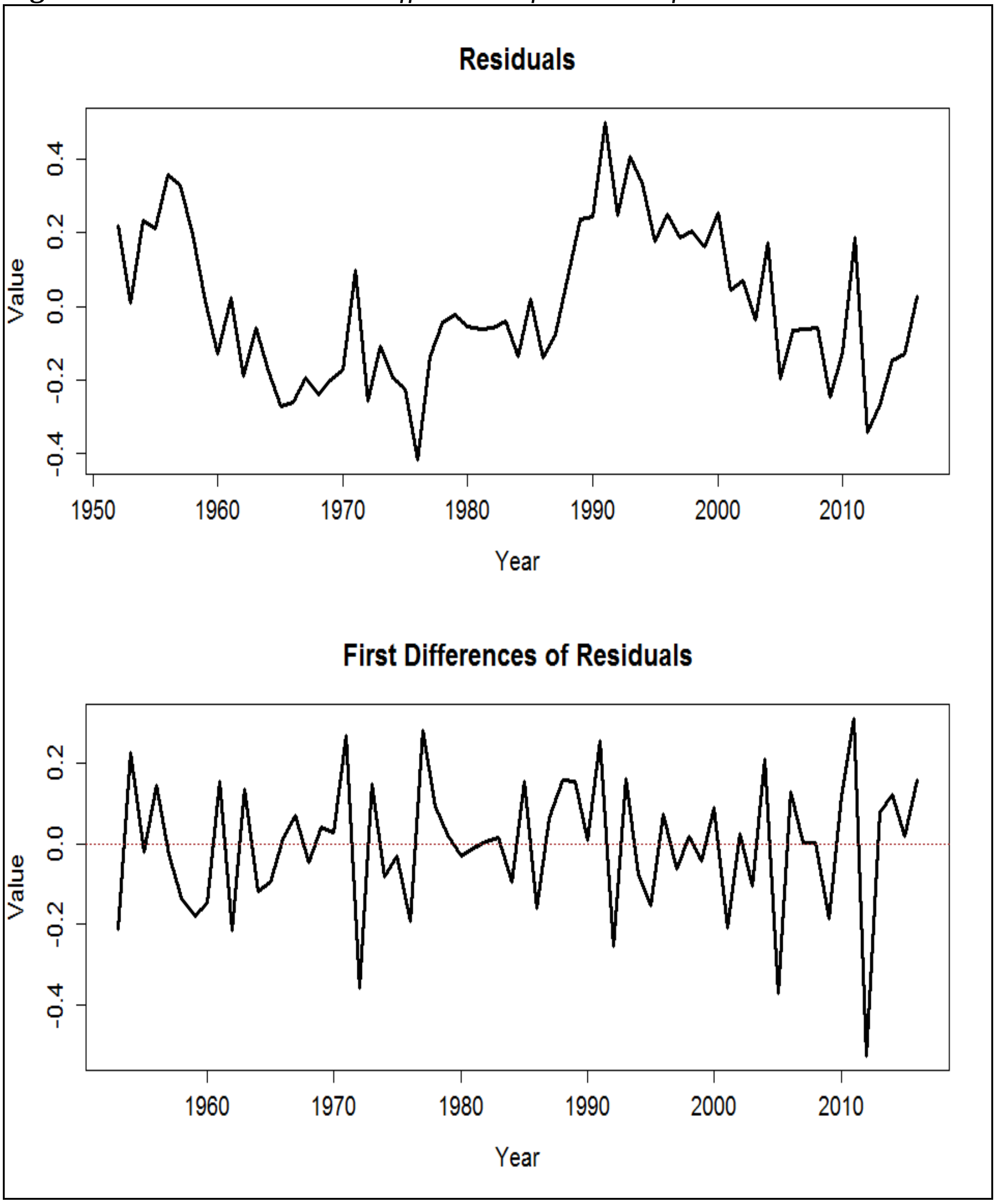
[18]See, e.g. Wooldridge (2013) for an introduction to OLS fitting.
[19]The OLS estimation is easily done in **R** with the **lm()** command.

Figure 6. *Long-Term Trend of Mortality Index*



An ARIMA model is then fit to the resulting residuals. As described in Section "Main Features of Time Series Analysis", we first investigate the residuals for stationarity. The graphical analysis of Figure 7 suggests that the residuals are not stationary, but their first differences might be.

Figure 7. *Residuals and First Differences of Residuals from the Model Fit*



This is confirmed statistically by the ADF test[20] and the ARCH-LM test[21]. The ARMA degrees are determined by the ACF and PACF, which are illustrated in Figure 8.

[20] $p - value \approx 0.019$.
[21] $p - value \approx 0.417$.

Figure 8. *ACF and PACF of First Differences of Residuals*

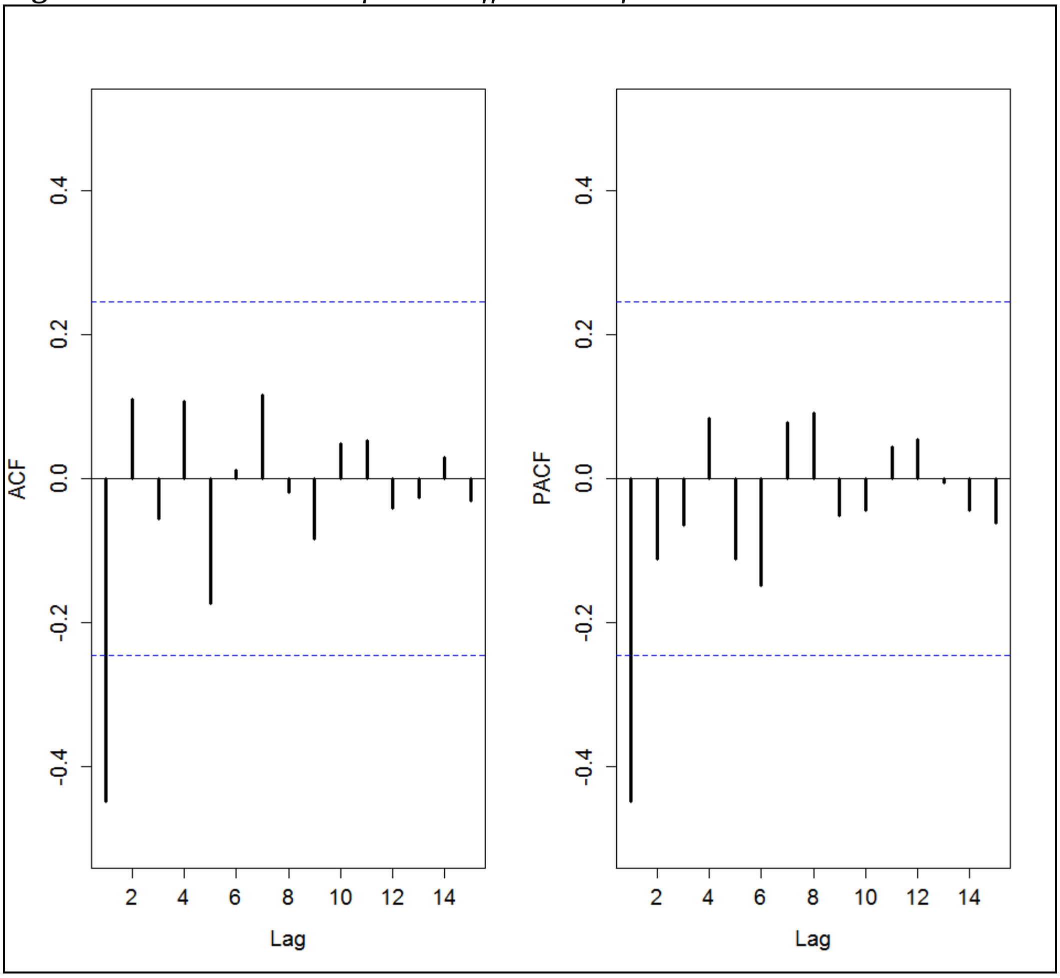


Figure 8 suggests that the first differences of the residuals might follow an AR(1) process with a negative coefficient, since the values alternate and decrease in tendency for the ACF, whereas they are almost zero after the first lag in the PACF. The OLS fit of an ARI(1,1) model to the residuals is highly significant and gives the model

$$(1 + 0.4584L)(1 - L)r_t = \varepsilon_t,$$

which becomes

$$r_t = 0.5416r_{t-1} + 0.4584r_{t-2} + \varepsilon_t, \tag{28}$$

where r_t is the residuum in period t and $\varepsilon_t \sim \mathcal{NID}(0, 0.1442^2)$. The combination of (27) and (28) yields

$$p_t = f(t) + r_t. \tag{29}$$

with p_t representing the PCs value in t. Equation (29) can then be used for simulation of the future development of the Mortality Index with Wiener Processes[22]. In the example, 10,000 future paths of the PCs are simulated.

The hypothetical history of the PCs is calculated through the matrix notation of equation (1):

$$\boldsymbol{P} = \boldsymbol{V}\boldsymbol{E} \tag{30}$$

Here, $\boldsymbol{P}$ is a matrix with t rows and s columns. t is the number of observed periods, and s is the number of TS. Consequently, $\boldsymbol{P}$ has dimensions of 65×4 in the example. $\boldsymbol{V}$ is the matrix of all logit-ASSSRs and therefore has the same structure as $\boldsymbol{P}$, i.e., a columnwise collection of all logit-ASSSR-TS. $\boldsymbol{E}$ is a matrix of columnwise EVs.

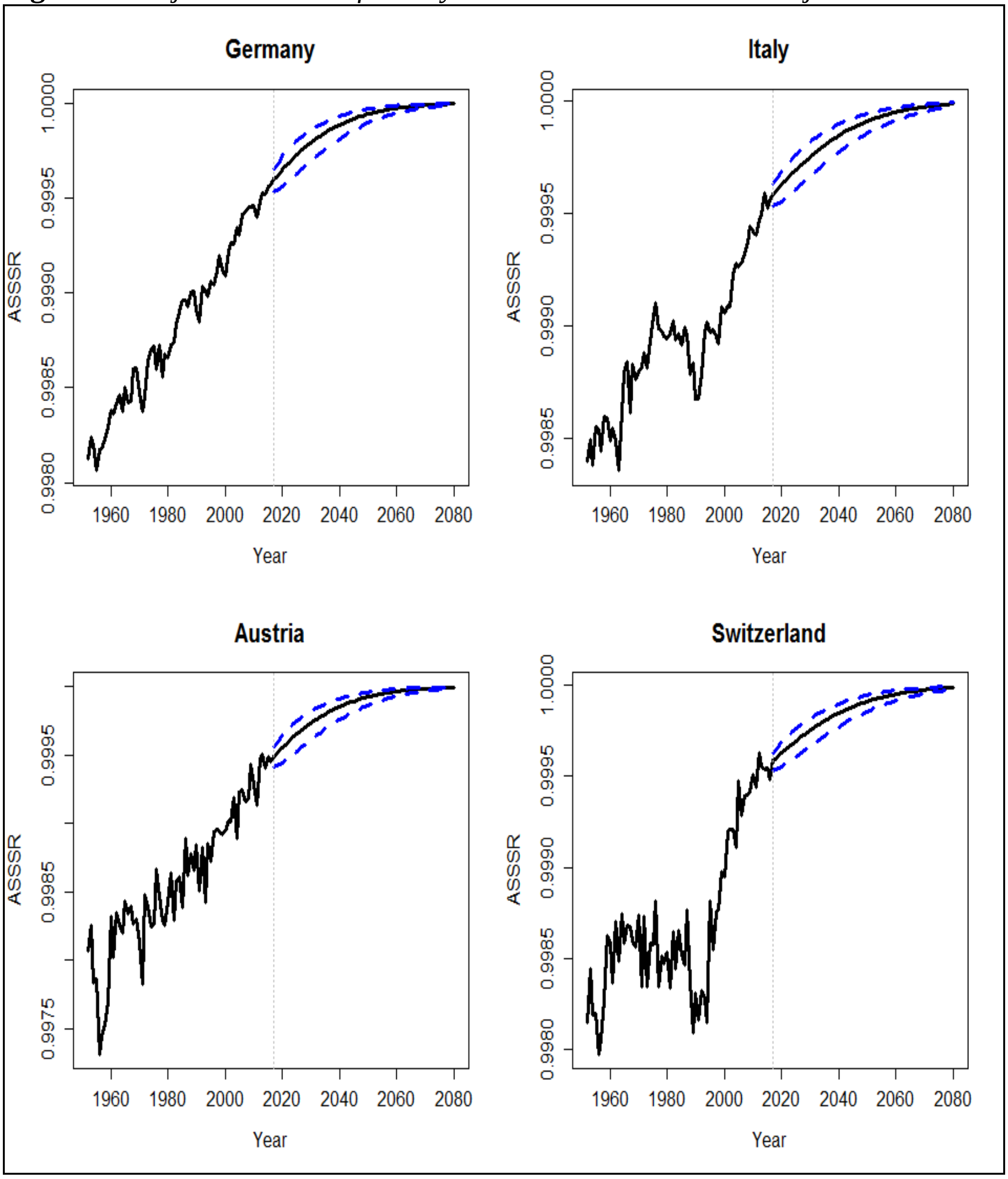
Through the reverse transformation of (30), the forecasts for the ASSSRs can be derived from the simulated future values of the PCs:

$$\mathbf{Y}_t = \mathbf{\Pi}_t \boldsymbol{E}^{-1}. \tag{31}$$

In this case, $\mathbf{\Pi}_t$ $(10{,}000 \times 4)$ is the simulation matrix[23] of the PCs in year t. $\boldsymbol{E}^{-1}$ (4×4) is the inverse of the eigenvector matrix, and the resulting matrix product $\mathbf{Y}_t$ is a $10{,}000 \times 4$ matrix of the simulation values for the logit-ASSSRs, estimated indirectly via PC simulation. Since the PCs are uncorrelated, simultaneous and independent computer simulations for each PC may be done separately to estimate PIs for the ASSSRs. A sufficiently large number of future trajectories must be estimated so that the PCs converge. The author performs 10,000 estimates to simulate the PCs until the year 2080. The simulation of PC 1 is based on (29). Empirical quantiles can be estimated for the PIs based on these trajectories. Finally, we need to inversely logistically transform the logit-ASSSRs to obtain the simulation values for ASSSRs and derive PIs from them. Figure 9 provides the 95% PIs for the ASSSRs for 25-year-old males in Germany, Italy, Austria, and Switzerland.

[22]See, e.g., Vanella (2017) on that.
[23]In the example, 10,000 simulations were conducted. Theoretically, this process works for a larger number of iterations, but the computation becomes cumbersome.

Figure 9. *Projected ASSSRs for 25-year-old Males with 95% Projection Intervals*



Sources: Destatis, 2005, 2015a, 2015b, 2015c, 2016, 2017a, 2017b, 2018a, 2018b; Eurostat, 2018; Human Mortality Database, 2018a, 2018b, 2018c, 2018d; Istat, 2018a, 2018b, 2018c, 2018d, 2018e, 2018f; STATcube, 2018; Own calculations and design.

We observe a quite similar future development, as could be expected by the high correlations among the ASSSRs. It should be stressed once more that this is just a simulation study, not a forecast. Counterintuitively, the PIs become narrower over time. In general, PIs will become wider, since uncertainty regarding the far future is greater than that regarding the near future. In this specific case, the result does make sense. Barring that landslide events such as wars or vast pandemics occur, mortality will on average decrease further. Since survival probabilities logically cannot become larger than one, it follows that in the very long term, survival rates will converge towards one in all scenarios, so the intervals become tighter.

Conclusion, Limitations and Outlook

The primary goal of this study has been the presentation of PCA and its practical implementation. On the basis of PCA, arbitrary age- and sex-specific measures, including the quantification of the stochasticity through PIs, may be modeled and forecast without bias, which was illustrated for the ASSSRs of 25-year-old males in Germany, Italy, Austria, and Switzerland. The presented approach is applied internationally on a country level; nevertheless, it may be applied on regional level as well if the required data are available. The example was only mortality for one age group to keep the paper concise, and mortality trends are the easiest expositions due to their clear trends in industrialized countries. Nevertheless, the methods presented can also be applied to other demographic phenomena (fertility, migration) or other fields (e.g., economics; meteorology), depending on the quality of available data.

PCA is a powerful tool for the simplification of complex phenomena and addressing correlation among different variables; however, PCA needs good data to work appropriately. Similar to all quantitative methods, PC forecasts with TSA models cannot address trends that have not been observed in the past. Therefore, forecasting should always assess the possibility of massive structural breaks occurring in the future. Moreover, a qualitative assessment of the PCs is advisable. A PC always represents a composition of the original variables. Therefore, an appropriate interpretation is very important, and PCA results have to be considered judiciously.

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