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A Stochastic Approach of Dynamical Systems by Means of Cocycles

Codruta Stoica

Abstract

As not all systems of differential equations that describe real world phenomena are deterministic, their approach has to combine the classic study with methods of stochastic analysis, which also provides sharp instruments for the study of deterministic ordinary infinite dimensional equations. A remarkable aspect is that of using the analytic method of the dynamical systems theory, such as the cocycles approach, in order to study the existence problems and long-time behavior for stochastic differential equations. The aim of this paper is to emphasize some dichotomic asymptotic behaviors in mean square for stochastic evolution cocycles. Our approach is based on the extension of some techniques from the deterministic framework constructed initially for skew-evolution semiflows on Hilbert spaces.

Keywords: Measurable projections, Stochastic dichotomic behaviors, Stochastic evolution cocycles.

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Introduction

The approach of the dynamical systems by means of associated operator families allows obtaining answers to some open problems by involving techniques of functional analysis and operator theory. To this purpose, the skew-evolution cocycles that we have defined in [8], are generalizations for evolution operators, semi-groups of operators and skew-product semi-flows. The asymptotic properties of skew-evolution semi-flows on infinite dimensional spaces were published in [9] - [13].

Several authors have focused their interest on the problem of the existence of stochastic semi-flows associated to stochastic evolution equations, as, for example, G. Da Prato and J. Zabczyk in [6] or F. Flandoli in [7]. For linear stochastic evolution equations with finite-dimensional noise, a stochastic semi-flow was obtained by A. Bensoussan and F. Flandoli in [2]. The property of non-uniform stability for stochastic differential equations is studied, for example, by G. Da Prato and A. Ichikawa in [5]. The exponential dichotomy in a stochastic setting was discussed by many authors, such as A.M. Ateiwi in [1] or T. Caraballo et al. in [4].

The problem of existence of stochastic semi-flows for the semi-linear stochastic evolution equation is a non-trivial one, mainly due to the well-established fact that the finite-dimensional methods for constructing, even continuous, stochastic flow break down in the infinite-dimensional setting of semi-linear stochastic evolution equations (see [14] and [15]).

The asymptotic behaviors in mean square for stochastic cocycles that we aim to emphasize in this paper are various classes of dichotomies, such as the exponential dichotomy in mean square, the exponential dichotomy of *(α,β)*-type in mean square, the polynomial dichotomy in mean square and the polynomial dichotomy of *(α,β)*-type in mean square. The *(α,β)*-type behaviors in mean square are generalizations of asymptotic properties given in [3].

Notations and Definitions

Let X be a real separable Hilbert space, $L(X)$ the set of all X-valued bounded operators defined on X, $B(X)$ the collection of all Borel sets on X, $T=\left\{(t,s)\in \mathbb{R}_+^2 \mid t\geq s\geq 0\right\}$. The norm of vectors on X and of operators on $L(X)$ is denoted $\|\cdot\|$ and the identity operator on X is I_X.

Let $(W, F, \{Ft\}_{t\geq 0}, P)$ be a standard filtered probability space, i.e. (W, F, P) is a probability space, $\{Ft\}_{t\geq 0}$ is an increasing family of σ-algebras such that F_0 contains all P-null sets of F and $F_t = \cap F_s$, $s \geq t$, for all $t \geq 0$. We will consider the set $Y = W\times X$.

Definition 2.1. The measurable random field $\varphi:(T\times W, B(X)\times F)\to(W,F)$ is called *stochastic evolution semiflow* if:

(s_1) $\varphi(t,t,w)=w$, $\forall (t,w)\in R_+\times W$;

(s_2) $\varphi(t,s,\varphi(s,t_0,w))=\varphi(t,t_0,w)$, $\forall(t,s),(s,t_0)\in T,\forall w\in W$.

Definition 2.2. The mapping $\Phi:(T\times W,B(X)\times F)\to L(X)$ is called *stochastic evolution cocycle* over an stochastic evolution semiflow φ if:

(c_1) $\Phi(t,t,w)=I_X$, $\forall(t,w)\in R_+\times W$;

(c_2) $\Phi(t,s,\phi(s,t_0,w))\Phi(s,t_0,w)=\Phi(t,t_0,w)$, $\forall(t,s),(s,t_0)\in T,\forall w\in W$.

Families of Measurable Projections

Let us consider a stochastic evolution cocycle Φ over a stochastic evolution semi-flow φ and let us suppose that the phase space X is decomposed, at every moment and for all $w\in W$, into two subspaces, i.e. $X = X_1(w) \oplus X_2(w)$. We denote by $\{Q_k(w)\}_{k=1,2}$ the family of measurable projections associated with the considered decomposition. The F-measurable subspaces $X_1(w)$ and $X_2(w)$ are called the *stable subspace*, respectively the *instable subspace*.

Definition 3.1. The subspaces $X_1(w)$ and $X_2(w)$ are said to be *invariant* relative to the stochastic evolution cocycle Φ if:

$$\Phi(t,s,w)X_k(w)\subset X_k(\varphi(t,s,w)),\ \forall(t,s)\in T,\forall w\in W,\ k=1,2.$$

Remark 3.2.

(i) The subspace $X_1(w)$ is finite dimensional with a fixed non-random dimension;

(ii) The subspace $X_2(w)$ is closed with a finite non-random co-dimension.

Definition 3.3. The family of measurable projections $\{Q_k(w)\}_{k=1,2}$ is called *compatible* with the stochastic evolution cocycle Φ if:

$$Q_k(\varphi(t,s,w))\Phi(t,s,w)=\Phi(t,s,w)Q_k(w),\ \forall(t,s)\in T,\forall w\in W.$$

Stochastic Dichotomic Behaviors

In what follows, we will give some asymptotic behaviors in mean square for a stochastic evolution cocycle Φ over a stochastic evolution semi-flow φ.

Definition 4.1. A stochastic evolution cocycle Φ is said to be *exponentially dichotomic in mean square (e.d.m.s.)* relative to a family of measurable projections $\{Q_k(w)\}_{k=1,2}$ compatible with Φ if there exist $N\geq 1$, $\alpha_1,\alpha_2>0$ such that the relations:

$$e^{\alpha_1 t}E\left\|\Phi(t,s,w)Q_1(w)x\right\|^2\leq Ne^{\alpha_1 s}E\left\|Q_1(w)x\right\|^2,$$

$$e^{\alpha_2 t}E\left\|Q_2(w)x\right\|^2\leq Ne^{\alpha_2 s}E\left\|\Phi(t,s,w)Q_2(w)x\right\|^2,$$

hold for all $(t,s)\in T$ and for all $(w,x)\in Y$.

Definition 4.2. A stochastic evolution cocycle *Φ* is said to be *(α,β)-exponentially dichotomic in mean square ((α,β)-e.d.m.s.)* relative to a family of measurable projections $\{Q_k(w)\}_{k=1,2}$ compatible with *Φ* if there exist $N \geq 1$, $\alpha_1,\alpha_2,\beta_1,\beta_2 > 0$ such that the relations:

$$e^{\alpha_1 t} E\left\|\Phi(t,s,w)Q_1(w)x\right\|^2 \leq Ne^{\beta_1 s} E\left\|Q_1(w)x\right\|^2,$$

$$e^{\alpha_2 t} E\left\|Q_2(w)x\right\|^2 \leq Ne^{\beta_2 s} E\left\|\Phi(t,s,w)Q_2(w)x\right\|^2,$$

hold for all $(t,s) \in T$ and for all $(w,x) \in Y$.

Definition 4.3. A stochastic evolution cocycle *Φ* is said to be *polynomially dichotomic in mean square (p.d.m.s.)* relative to a family of measurable projections $\{Q_k(w)\}_{k=1,2}$ compatible with *Φ* if there exist $N \geq 1$, $\alpha_1,\alpha_2 > 0$ such that the relations:

$$t^{\alpha_1} E\left\|\Phi(t,s,w)Q_1(w)x\right\|^2 \leq Ns^{\alpha_1} E\left\|Q_1(w)x\right\|^2,$$

$$t^{\alpha_2} E\left\|Q_2(w)x\right\|^2 \leq Ns^{\alpha_2} E\left\|\Phi(t,s,w)Q_2(w)x\right\|^2,$$

hold for all $(t,s) \in T$ and for all $(w,x) \in Y$.

Definition 4.4. A stochastic evolution cocycle *Φ* is said to be *(α,β)-polynomially dichotomic in mean square ((α,β)-p.d.m.s.)* relative to a family of measurable projections $\{Q_k(w)\}_{k=1,2}$ compatible with *Φ* if there exist $N \geq 1$, $\alpha_1,\alpha_2,\beta_1,\beta_2 > 0$ such that the relations:

$$t^{\alpha_1} E\left\|\Phi(t,s,w)Q_1(w)x\right\|^2 \leq Ns^{\beta_1} E\left\|Q_1(w)x\right\|^2,$$

$$t^{\alpha_2} E\left\|Q_2(w)x\right\|^2 \leq Ns^{\beta_2} E\left\|\Phi(t,s,w)Q_2(w)x\right\|^2,$$

hold for all $(t,s) \in T$ and for all $(w,x) \in Y$.

Connectors between Stochastic Dichotomic Behaviors

Obvious connections between the considered dichotomic properties in mean square are given by the following remarks.

Remark 5.1. An exponentially dichotomic stochastic evolution cocycle *Φ* is *(α,β)*-exponentially dichotomic.

Remark 5.2. A polynomially dichotomic stochastic evolution cocycle *Φ* is *(α,β)*-polynomially dichotomic.
The main results of this section are given in the next

Proposition 5.3. *An exponentially dichotomic stochastic evolution cocycle Φ is polynomially dichotomic.*
Proof. According to Definition 4.1, there exist $N \geq 1, \alpha_1 > 0$ such that

$$E\left\|\Phi(t,s,w)Q_1(w)x\right\|^2 \leq Ne^{-\alpha_1(t-s)}E\left\|Q_1(w)x\right\|^2,$$

for all $(t,s)\in T$ and $(w,x)\in Y$. Following inequalities hold

$$e^{-u} \leq \frac{1}{u+1}, \forall u \geq 0 \quad \text{and} \quad \frac{t}{s} \leq t-s+1, \forall t \geq s \geq 1.$$

We obtain that

$$E\left\|\Phi(t,s,w)Q_1(w)x\right\|^2 \leq N(t-s+1)^{-\alpha_1}E\left\|Q_1(w)x\right\|^2 \leq Nt^{-\alpha_1}s^{\alpha_1}E\left\|Q_1(w)x\right\|^2$$

for all $t \geq s \geq 1$ and $(w,x)\in Y$.

According to the same Definition 4.1, there exist $N \geq 1, \alpha_2 > 0$ such that

$$E\left\|Q_2(w)x\right\|^2 \leq Ne^{-\alpha_2(t-s)}E\left\|\Phi(t,s,w)Q_2(w)x\right\|^2,$$

As following inequality holds

$$\frac{e^s}{e^t} \leq \frac{s}{t}, \forall t \geq s > 0,$$

we have that

$$E\left\|Q_2(w)x\right\|^2 \leq Ne^{-\alpha_2 t}e^{\alpha_2 s}E\left\|\Phi(t,s,w)Q_2(w)x\right\|^2 \leq$$
$$\leq Nt^{-\alpha_2}s^{\alpha_2}E\left\|\Phi(t,s,w)Q_2(w)x\right\|^2$$

for all $t \geq s \geq 1$ and all $(w,x)\in Y$. It follows that the stochastic evolution cocycle Φ is polynomially dichotomic.

Proposition 5.4. *An (α,β)-exponentially dichotomic stochastic evolution cocycle Φ, with* $\alpha_1 \geq \beta_1$ *and* $\alpha_2 \geq \beta_2$*, is (α,β)-polynomially dichotomic.*

Proof. According to Definition 4.3, there exist $N \geq 1, \alpha_1 > 0$ such that

$$t^{\alpha_1}E\left\|\Phi(t,s,w)Q_1(w)x\right\|^2 \leq Ns^{\alpha_1}E\left\|Q_1(w)x\right\|^2,$$

for all $(t,s)\in T$ and $(w,x)\in Y$. Similarly as in the proof of Proposition 5.3, for $\alpha_1 \geq \beta_1$, we have

$$E\left\|\Phi(t,s,w)Q_1(w)x\right\|^2 \leq Nt^{-\alpha_1}e^{-\beta_1 s}s^{\beta_1}e^{\beta_1 s}E\left\|Q_1(w)x\right\|^2 =$$
$$= Nt^{-\alpha_1}s^{\beta_1}E\left\|Q_1(w)x\right\|^2$$

for all $(t,s)\in T$ and $(w,x)\in Y$.

Again, according to Definition 4.3, there exist $N \geq 1, \alpha_2 > 0$ such that

$$t^{\alpha_2}E\left\|Q_2(w)x\right\|^2 \leq Ns^{\alpha_2}E\left\|\Phi(t,s,w)Q_2(w)x\right\|^2,$$

for all $(t,s)\in T$ and $(w,x)\in Y$. In a similar way, we obtain for $\alpha_2 \geq \beta_2$

$$E\left\|Q_2(w)x\right\|^2 \leq Ne^{-\alpha_2 t}e^{-\beta_2 t}E\left\|\Phi(t,s,w)Q_2(w)x\right\|^2 \leq$$

$$\leq Nt^{-\alpha_2}e^{-\beta_2}E\left\|\Phi(t,s,w)Q_2(w)x\right\|^2$$

for all $(t,s)\in T$ and $(w,x)\in Y$. We obtain that the stochastic evolution cocycle Φ is polynomially dichotomic, and, further, according to Remark 5.2, (α,β)-polynomially dichotomic.

Remark 5.5. The connections between the dichotomic behaviors in mean square of a stochastic cocycle are given in the next diagram

$$\begin{array}{ccc} \text{e.d.m.s.} & \Rightarrow & \text{p.d.m.s.} \\ \Downarrow & & \Downarrow \\ (\alpha,\beta)\text{-e.d.m.s.} & \Rightarrow & (\alpha,\beta)\text{-p.d.m.s.} \end{array}$$

Conclusions

It is well known that in the qualitative theory of evolution equations, either it is deterministic or not, the exponential dichotomy is one of the most important asymptotic properties. Also, it is worth mentioning that the approach of dynamical systems in cases where randomness is involved is more appropriate to be done by means of stochastic cocycles. In order to give an answer to both issues, we propose a general framework for the study of several stochastic dichotomic behaviors for the random dynamical systems.

References

[1] Ateiwi, A.M. 2002. 'About bounded solutions of linear stochastic Ito systems', *Miskolc Math. Notes*, 3:3-12.

[2] Bensoussan, A. Flandoli, F. 1995. 'Stochastic inertial manifold', *Stoch. and Stoch. Reports*, 53:13-39.

[3] Barreira, L., Valls, C. 2008. 'Stability of Nonautonomous Differential Equations', *Lecture Notes in Mathematics* 1926.

[4] Caraballo, T., Duan, J., Lu, K., Schmalfuss, B. 2010. 'Invariant manifolds for random and stochastic partial differential equations', *Adv. Nonlinear Stud.*, 10:23-52.

[5] Da Prato, G., Ichikawa, A. 1987. 'Lyapunov equations for time-varying linear systems', *Systems Control Lett.,* 9:165-172.

[6] Da Prato, G., Zabczyc, J. 1992. 'Stochastic Equations in Infinite Dimensions', University Press, Cambridge.

[7] Flandoli, F. 1996. 'Stochastic flows for nonlinear second-order parabolic stochastic partial differential equations', '*Ann. Probab*., 24:547-558.

[8] Megan, M., Stoica, C. 2008. 'Exponential instability of skew-evolution semiflows in Banach spaces', *Studia Univ. Babes-Bolyai Math*., LIII(1):17-24.

[9] Stoica, C. 2011. 'Uniform asymptotic behaviors for skew-evolution semiflows on Banach spaces', Mirton Publishing House, Timisoara.

[10] Stoica, C. 2013. ‘Various concepts of trichotomy for evolution equations on Banach spaces, *Essays on Mathematics and Statistics*, Vol. 3:215-224.
[11] Stoica, C. 2015. ‘Exponential dichotomy versus exponential splitting for skew-evolution semiflows in infinite dimensional spaces’, *ATINER’s Conference Paper Series*, No: FOR2015-1744: 3-9.
[12] Stoica, C. 2015. ‘Cocycles over non autonomous dynamical systems in Banach spaces’, *Essays on Mathematics and Statistics*, Vol. 5:65-72.
[13] Stoica C., Megan M. 2010. ‘On uniform exponential stability for skew-evolution semiflows on Banach spaces’, *Nonlinear Anal.*, 72(3-4):1305-1313.
[14] Stoica, D. 2010. ‘Uniform exponential dichotomy of stochastic cocycles’, *Stochastic Process. Appl.*, 120:1920-1928.
[15] Stoica D., Megan M. 2012. ‘On nonuniform dichotomy for stochastic skew-evolution semiflows in Hilbert space’, *Czechoslovak Math. J.*, 62(137):879-887.