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Markovian Algorithm for Vanilla Option Valuation

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Abstract

To price and hedge derivative securities, it is crucial to have a good model of the probability distribution of the underlying product. The most famous continuous time model is the celebrated Black - Scholes model, which uses the normal distribution to fit the log returns of the underlying asset. One of the main problems with this model is that the data suggest that the log returns of stocks are not normally distributed. So other more flexible distributions are needed. In the article is proposed a numerical model how to model dynamics of asset prices by Markov process in continuous time with countable set of states based on phase type distribution.

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1 Introduction

One of the main questions facing quantitative finance is how best to model the dynamics of prices in financial markets. Armed with the model of price dynamics, an investor can calculate theoretical prices for derivative securities.

Applications in financial mathematics have relied heavily on diffusion process for each underlying asset price – that is, a stochastic integral or stochastic differential equation where the uncertainty is driven by Brownian motion, with a constant volatility. This implies a normal distribution for the continuously - compound stock return, or lognormal distribution for the stock prices [14] . Unfortunately, it appears that asset prices in the real world are not driven by diffusion processes. The log returns of most financial assets do not follow a normal law. They are skewed and have an actual kurtosis higher than that of the normal distribution. There are more important reasons why the above mentioned approach is not adequate. Examples include fat tails, volatility clustering, large discrete jumps, parameter instability, and asymmetric correlations [1, 2]. The underlying normal distribution was replaced by a more sophisticated one. Examples of such, which can take into account skewness, excess kurtosis and other features, are the Variance Gamma [3], the Normal Inverse Gaussian [4], the CGMY (named after Carr, Geman, Madan and Yor) [5], the Hyperbolic Model [6] and the Meixner [7] distribution. Including such features makes analytic modelling less tractable, and potentially makes numerical modelling a more attractive alternative. In the following sections the algorithm of modelling asset prices by Markov chains is proposed.

2 Markov Property of Stock Prices

The dynamics of asset prices are reflected by uncertain movements of their values over time. Some authors [8, 9] state that Efficient Market Hypothesis (EMH) is one possible reason for the random behaviour of the asset price. The EMH basically states that past history is fully reflected in present prices and markets respond immediately to new information about the asset.

A Markov process is a particular type of stochastic process where only the present value of a variable is relevant for predicting the future. The past history of the variables and the way that that the present has emerged from the past are irrelevant.

Stock prices are usually assumed to follow a Markov process. These processes are important models of security prices, because they are often realistic representation of true prices and yet the Markov property leads to simplified computations. If the stock price follows a Markov process, our predictions of the future should be unaffected by the price one week ago, one month ago, or one year ago. The only relevant piece of information is the price now. Predictions are uncertain and must be expressed in terms of probability distributions. The Markov property implies that the probability distribution of the price at any particular future time is not dependent on the particular path followed by the price in the past.

If stock price process $S = \{S_t, 0 \le t \le T\}$ is markovian and if denotes by $F = \{F_t, 0 \le t \le T\}$ the natural filtration of S (intuitively, F_t contains all market information up to time t), then we can write for a well-behaved function *f:* $E[f(S_{T)} \setminus \mathrm{F}_t] = E[f(S_T) \setminus S_t]$. The stock price process takes values in some countable set *E*, called the state space. If $S_t = j \in E$, we shall say 'the process is in state *j* at time *t*'. The most common situation is for the state to be a scalar, but frequently it is more convenient for the state to be a vector.

3 Constructing the Probability Distribution of Stock Price Movement

Our aim is to construct the stock price dynamics as Markov process with continuous time and countable space of states. To find the space of states and transition rates between them we have to construct price movement distributions up and down for a given stock. To get Markov process the distribution of time length of stock price rising or decreasing must be exponential with parameter μ. Unfortunately, usually it is insufficient, then a convenient representation for more general distributions is the Coxian formulation [10]. This formulation, by means of fictitious phases, allows the duration of generating stock price rate of transition up or down to be described by a linear combination of stochastic variables. Thus, generation of price movement is a continuous succession of *k* phases, each having exponential service time distribution of rate μ_j, $j = 1,2,\cdots,k$. After phase *j*, a stock price leaves the phases with probability ($1 - p_j$). The stock price can occupy only one phase at a time. Therefore, there can be at most one stock price within the set of phases at any time.

Let us consider a general probability distribution function *G*(*t*) of stock price movement. Useful approximation of this function can be obtained by the mixture and convolutions of exponential (phase-type) distributions. Then a Markov chain with a countable space of states and continuous time can represent the evolution of stock price dynamics. Suppose we let m_k, $k = \overline{1,3}$ denotes the *k*th non-central moment, i.e. *E* [X^k], where *X* is a random variable of price movement time. Define a random variable *X* in this way:

$$X = \begin{cases} X_1 & \text{with prob.} \quad p_2; \\ X_1 + X_2 & \text{with prob.} \quad p_1, \end{cases} \qquad 1)$$

where X_1 and X_2 are independent random variables having exponential distribution with parameters μ_1 and $p_2\mu_2$ respectively; $p_1 + p_2 = 1$. It is easy to verify that the density function of *X* is given by

$$f(x) = p_2\mu_1\left(\frac{\mu_2 - \mu_1}{\mu_2 p_2 - \mu_1} e^{-\mu_1 x} - \frac{\mu_2 p_1}{\mu_2 p_2 - \mu_1} e^{-p_2\mu_2 x}\right) \qquad (2)$$

Note. Duration of service time as a random variable given by (1) allows us to apply the method for automatic construction of numerical models for systems described by Markov process [13].

Moment matching is a common method for approximating distributions [11, 12]. Though two-moment approximations are common, they may lead to serious error when the coefficient of variation ν, (the standard deviation divided by the mean) is high. The first three moments of any non degenerate distribution with support on [0,∞) can be matched by the distribution (2).

To obtain the values of the parameters μ_1, μ_2, p_1 and p_2 of approximation, a complex system of non-linear equations needs to be solved:

$$\begin{cases} \dfrac{1! \cdot p_2 \mu_1}{\mu_2 p_2 - \mu_1} \left(\dfrac{\mu_2 - \mu_1}{\mu_1^2} - \dfrac{\mu_2 p_1}{\mu_2^2 p_2^2} \right) = m_1; \\ \dfrac{2! \cdot p_2 \mu_1}{\mu_2 p_2 - \mu_1} \left(\dfrac{\mu_2 - \mu_1}{\mu_1^3} - \dfrac{\mu_2 p_1}{\mu_2^3 p_2^3} \right) = m_2; \\ \dfrac{3! \cdot p_2 \mu_1}{\mu_2 p_2 - \mu_1} \left(\dfrac{\mu_2 - \mu_1}{\mu_1^4} - \dfrac{\mu_2 p_1}{\mu_2^4 p_2^4} \right) = m_3; \\ p_1 + p_2 = 1. \end{cases} \tag{3}$$

The solution of the system is the following:

$$\mu_2 = \frac{g_2 - g_1^2}{g_1^3 - 2g_1 g_2 + g_3}, g_k = \frac{m_k}{k!}, k = \overline{1,3};$$

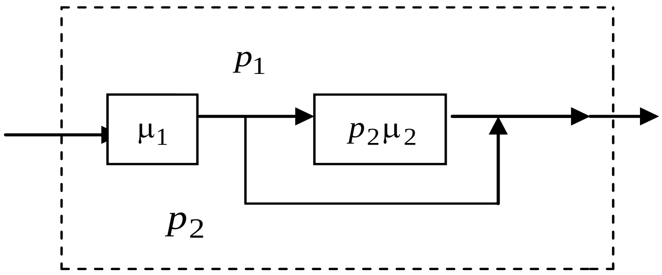
$$\mu_1 = \frac{1 + \mu_2 g_1 \pm \sqrt{(1 - \mu_2 g_1)^2 + 4\mu_2^2 (g_2 - g_1^2)}}{2g_1 - 2\mu_2 (g_2 - g_1^2)};$$

(4)

$$p_1 = \frac{\mu_2 (\mu_1 g_1 - 1)}{\mu_2 (\mu_1 g_1 - 1) + \mu_1}; \quad p_2 = \frac{\mu_1}{\mu_2 (\mu_1 g_1 - 1) + \mu_1}.$$

The exponential stages are shown graphically in Fig.1.

Figure 1. The diagram of two exponential phases



4 State Space Representation and Transitions

In this section we will use so called the event language to generate the space of stock prices and transition matrix between them. We approximate the distributions of price movement time up and down from the historical data of selected stock prices by the formulas respectively

$$f_u(y) = p_1^u \mu_1^u \left(\frac{\mu_2^u - \mu_1^u}{\mu_2^u p_2^u - \mu_1^u} e^{-\mu_1^u y} - \frac{\mu_2^u p_2^u}{\mu_2^u p_2^u - \mu_1^u} e^{-\mu_2^u p_2^u y} \right)$$

$$f_d(y) = p_1^d \mu_1^d \left(\frac{\mu_2^d - \mu_1^d}{\mu_2^d p_2^d - \mu_1^d} e^{-\mu_1^d y} - \frac{\mu_2^d p_2^d}{\mu_2^d p_2^d - \mu_1^d} e^{-\mu_2^d p_2^d y} \right)$$

A Markov chain with the countable space of states and continuous time can describe the dynamics of stock price movement. To construct a numerical model of the system, the approach proposed in [13] will be applied.

The set of events in the system:

$$E = \{e_1^u, e_2^u, e_3^u, e_4^u, e_1^d, e_2^d, e_3^d, e_4^d\}$$

where

$e_1^{u(d)}$ – beginning of price movement up (down);

$e_2^{u(d)}$ – completed the stage of price movement up (down) with probability $p_1^{u(d)}$ in the first phase;

$e_3^{u(d)}$ – completed the stage of price movement up (down) with probability $p_2^{u(d)}$ in the first phase;

$e_4^{u(d)}$ – completed the stage of price movement up (down) in the second phase;

The set of transition rates:

$Intens = \{\mu_1^u, p_2^u \mu_2^u, \mu_1^d, p_2^d \mu_2^d\}$,

where

$\mu_1^{u(d)}$ - completion rate of price movement up (down) in the first phase;

$\mu_1^{u(d)} p_2^{u(d)}$ - completion rate of price movement up (down) in the second phase;

Let us consider an asset observed on a discrete time scale $\{0,1,\cdots,t,\cdots,T\}, T < \infty$ having $S(t)$ as market stock value at time t. To model the basic stochastic process $(S(t), t = 0,1,\cdots,T)$ we suppose that the asset has known minimal and maximal values so that the set of all possible values is the closed interval $[S_{\min}, S_{\max}]$ [15]. For example, if S_0 is the value of the asset at time 0, we can put

$$S_0 = (S_{\max} + S_{\min})/2$$
$$S_k = S_0 + k\Delta, k = 1,\cdots,n$$
$$S_{-k} = S_0 - k\Delta, k = 1,\cdots.n$$
$$\Delta = (S_{\max} - S_{\min})/2n$$

n being chosen arbitrarily. This implies the total number of states is $2n + 1$. In what follows, we order these states in the naturally increasing order and use the following notation for the state space:

$$I = \{-n, -(n-1), \cdots, 0, 1, \cdots, n\}.$$

We can also introduce different step lengths following movements up and down and so consider respectively Δ, Δ'. It is also possible to let $S_{\max} \to \infty$ and $T \to \infty$ particularly to get good approximation results.

To model the dynamics of stock prices we need know the number of the phase in which is the stock price. The state space of the system is completely specified by the set of triples $B = \{(b_1, b_2, b_3)\}, b_1 \in I$,

where

$$b_2 = \begin{cases} 0, \text{if the stock price is not changing up;} \\ 1, \text{if the stock price is moving up in the first phase;} \\ 2, \text{if the stock price is moving up in the second phase.} \end{cases}$$

$$b_3 = \begin{cases} 0, \text{if the stock price is not changing down;} \\ 3, \text{if the stock price is moving down in the first phase;} \\ 4, \text{if the stock price is moving down in the second phase.} \end{cases}$$

The dynamics of stock price movement can be described in the event language. As an example, the description of the fourth event is represented bellow.

e_4^u : **if** $b_2 = 2$ **and** $b_1 < n$
then $b_1 \leftarrow b_1 + 1; b_2 \leftarrow 0$
Intense $\leftarrow \mu_1^u p_1$
end then
end if
end e_2^u

For creation of software for automatic construction of numerical models was applied the algorithm proposed in [13]. The software consists of:

- The language of a model specification
- A program for automatic generation of all possible states (the set of stock prices) and transition rates among them
- A program for calculation of steady state probabilities of Markov process
- A program for calculation performance measures.

The states of Markov process are generated from an initial state. All possible transitions from this state are considered. When this step is completed, the current state is marked, and one of the newly obtained states becomes the current state. The generation process terminates when all the states in the list have been marked and no new state is obtained. The stationary probabilities of the states are calculated solving the system of Chapman – Kolmogorov differential equations when the derivatives equal zero. The probabilities of stock price states are calculated by the following formula:

$$P(k) = \sum_{b_2, b_3} \pi(k, b_2, b_3), k = -n, \cdots, n,$$

where $\pi(k, b_2, b_3)$ is the probability of the price at the fixed phase.

5 Evaluation of European call option

Let us, that we want to evaluate call option with strike price K and expiry of T. Today's date is *t*. The option contract can be as European as well as American type. The payoff function for a European call at moment T is $C(T) = \max\{0, S(T) - K\}$. American call option at expiry is worth $\max_{t=1,\ldots,T}\{\max\{0, S_t - K\}\} = \max\{0, S_\tau - K\}$.

A risky security process is defind on the probability space $\{\Omega, F, P\}$. The given space is discrete and finite. Consider homogenuous Markov chain with the transition matrix $P = (p_{ij}),\ i, j = 1,...,2n+1$ defind on state space *S*. To describe the distribution of the option value at maturity we will follow the approach represented in [15]. Denote at the moment t the value of the option by $C(t)$ and the stock price by $S(t) = S_t$. The life time of the option equal to $T - t$ with strike price $K = k_0\Delta$. Then the distribution of the option value at expiry T is given by

$$\begin{aligned} P[C(T) = (j - k_0)\Delta] &= p_{S_t,j}^{(T-t)}, \quad j > k_0 \\ P[C(T) = 0] &= \sum_{l \le k_0} p_{S_t,l}^{(T-t)}, \quad j \le k_0 \end{aligned} \tag{4}$$

In view of (4), it is possible to compute various probablistic characteristics of the option value C. For instance, the conditional mean of $C(T)$ equal to

$$E[C(T) \,|\, S(t) = S_t] = \sum_{j>k_0} p_{S_t,j}^{(T-t)} (j - k_0)\Delta \tag{5}$$

The value of option at moment t with the given risk free interest rate i is

$$\begin{aligned} &C(t) = v^{(T-t)} E[C(T) \,|\, S(t) = S_t] = v^{(T-t)} \sum_{j>k_0} p_{S_t,j}^{(T-t)} (j - k_0)\Delta; \\ &v = (1+i)^{-1}; \end{aligned} \tag{6}$$

If the transition matrix P is ergodic and the option duration T-t is quite large then the the expressions (4), (5) ir (6) can be rewritten in the following form:

$$\left.\begin{aligned} P[C(T) = (j - k_0)\Delta] &= \pi_j, \quad j > k_0 \\ P[C(T) = 0] &= \sum_{l \le k_0} \pi_l, \quad j \le k_0 \end{aligned}\right\} \tag{7}$$

$$\left.\begin{aligned} &E[C(T) \,|\, S(t) = S_t] = \sum_{j>k0} \pi_j (j - k_0)\Delta \\ &C(t) = v^{(T-t)} \sum_{j>k_0} \pi_l (j - k_0)\Delta \end{aligned}\right\} \tag{8}$$

where $\pi = (\pi_1, ..., \pi_{2n+1})$ is the stationary distribution of the states.

6 Numerical examples

Consider the historical prices of „*Microsoft Corporation*" with data window equal to four months. The analysis of prices showed that $S_{\max} = 28{,}10, \mathrm{S}_{\min} = 24{,}15$. The mean daily increament of stock price is $\Delta = 0{,}1796$. The prices of the stock was divided into 22 intervals. From the time series are obtained the following estimations of initial moments:

$$\begin{aligned} m_1^U &= 2{,}2402; m_2^U = 11{,}6950; m_3^U = 85{,}7250; \\ m_1^D &= 1{,}8321; m_2^D = 6{,}9676; m_3^D = 37{,}4190. \end{aligned}$$

The parameters of the approximation are

$$\begin{aligned} \mu_1^U &= 0{,}4014; \mu_2^U = 1{,}2384; \mathrm{p}_1^{\mathrm{U}} = 0{,}2373; \\ \mu_1^D &= 0{,}5381; \mu_2^D = 0{,}3358; \mathrm{p}_1^{\mathrm{D}} = 0{,}0088. \end{aligned}$$

The suggested method allows to describe the dynamics of stock prices by Markov process. The stock prices intervals represents the state space. The suggested algorithm was realized in programming language C++.

Historical stock prices of McDonalds were analysed over last four years. Consider the European call option under the created model with the following parameters: S = 52,26, i = 5% ant T = 10. Several option prices with different strike prices were evaluated. The results are presented in the table 1.

Table.1. Prices of call option

Strike price	The expected vakue	Price of option
35,34	38,42	23,59
45,99	10,31	6,33
50,38	5,92	3,63
55,39	3,08	1,89
60,41	1,54	0,95
65,42	0,72	0,44
75,45	0,05	0,03

7 Conclusion

Empirical investigation showed that more than half of analyzed log returns of stocks are not normally distributed. So in the paper is proposed a new approach how unknown distribution of stock prices approximate by the mixture of exponential distributions and then describe the dynamics of stock prices by Markov process with countable state space and continuous time. The created software allows according described events in the system automatically generate the state space and the matrix of transition probabilities. Having stationary distribution of stock prices it is possible to evaluate option price under investigation. The main interest of this model is that it works even when there are possibilities of arbitrage, i.e. the most frequent cases.

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